FACTORIZATION AND RESUMMATION BEYOND LEADING POWER

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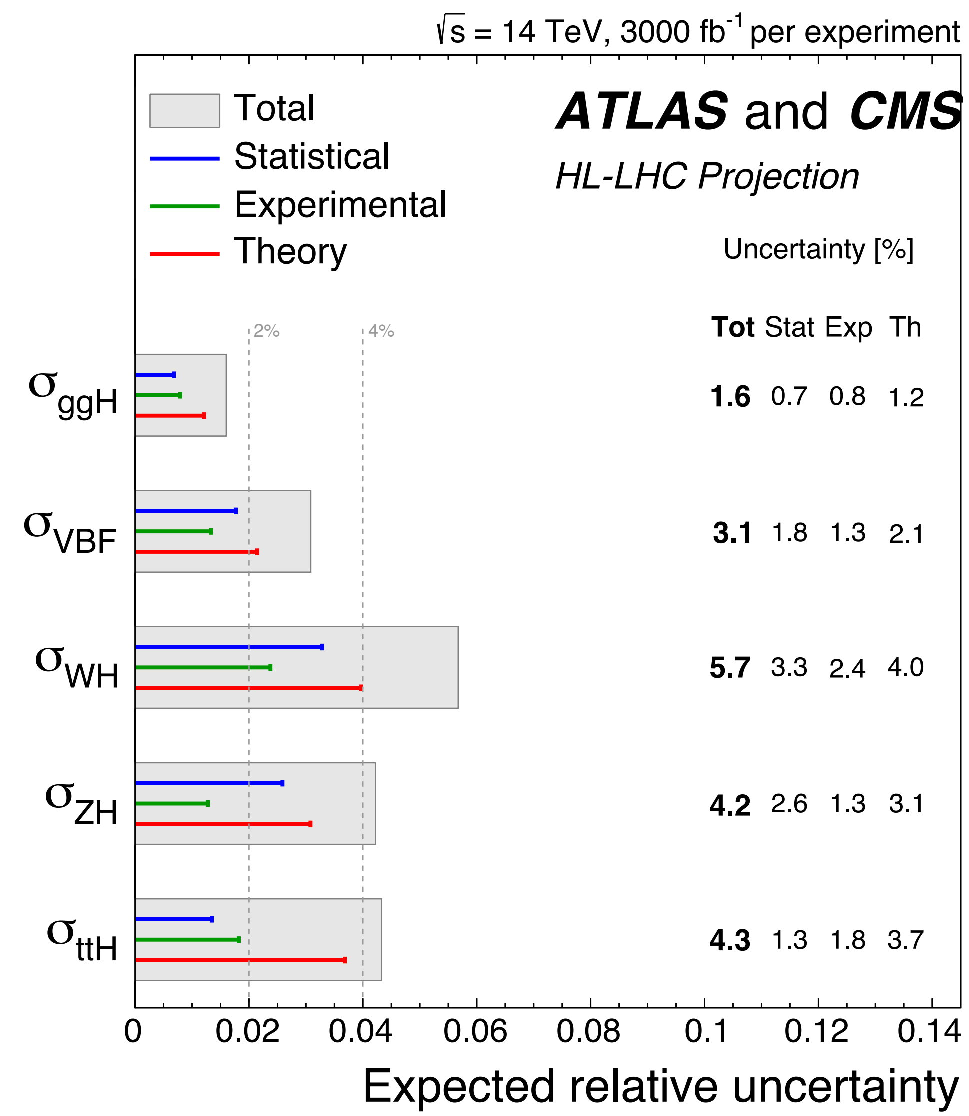
PRECISION PHYSICS AT THE LHC

- 1% precision goal for LHC observables
- Major increase in statistics in Run III and the High-Luminosity phase of the LHC
- Future measurements limited by theoretical accuracy



[Figure from Working Group on Higgs Physics at the HL-LHC, 1902.00134]

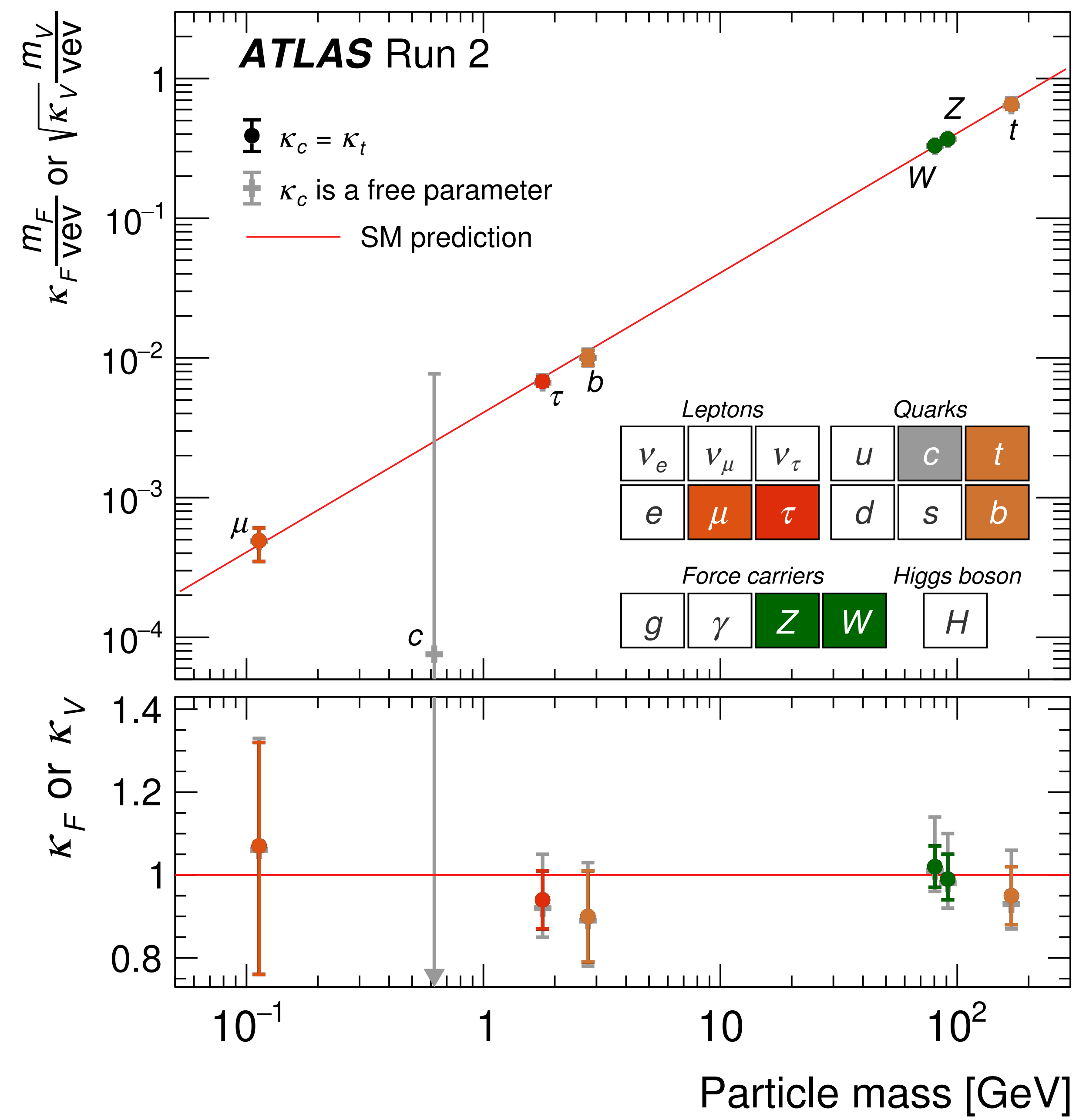
Tree level: classical field theory

Loop corrections: quantum effects

→ **Uncover the structure of the theory to understand physics**

HIGGS COUPLINGS

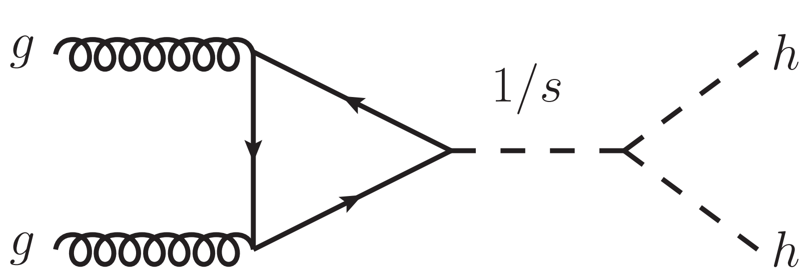
- Top-quark Yukawa coupling first directly observed in 2018 ($t\bar{t}H$)
  [ATLAS Collaboration, 1806.00425] [CMS Collaboration, 1804.02610]
- Higgs coupling to second generation leptons measured in 2020 ($H \to \mu^+\mu^-$)
  [ATLAS Collaboration, 2007.07830] [CMS Collaboration, 2009.04363]
- Higgs coupling to second generation quarks measured in 2022 ($H \to c\bar{c}$)
  [ATLAS Collaboration, 2201.11428] [CMS Collaboration, 2205.05550]



[ATLAS Collaboration, CERN-EP-2022-057, 2207.00092]

- The Higgs self-coupling is yet to be measured

$$V(H) = \frac{1}{2}m_H^2 H^2 + \lambda v H^3 + \frac{\lambda}{4}H^4 \qquad m_H^2 = 2\lambda v^2$$



PERTURBATIVE EXPANSION

Resummation logarithmic order

Loop order

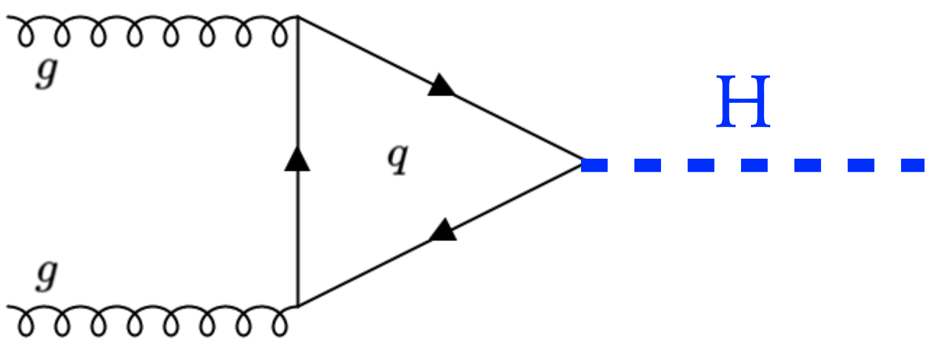
	LL	NLL	NLL’	N2LL	N2LL’	N3LL	N3LL’
LO	1						
NLO	$\alpha_s L^2$	$\alpha_s L$	α_s				
NNLO	$\alpha_s^2 L^4$	$\alpha_s^2 L^3$	$\alpha_s^2 L^2$	$\alpha_s^2 L$	α_s^2		
N3LO	$\alpha_s^3 L^6$	$\alpha_s^3 L^5$	$\alpha_s^3 L^4$	$\alpha_s^3 L^3$	$\alpha_s^3 L^2$	$\alpha_s^3 L$	α_s^3
N4LO	$\alpha_s^4 L^8$	$\alpha_s^4 L^7$	$\alpha_s^4 L^6$	$\alpha_s^4 L^5$	$\alpha_s^4 L^4$	$\alpha_s^4 L^3$	$\alpha_s^4 L^2$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

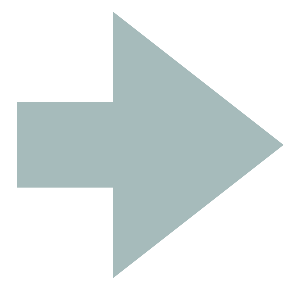
$$\frac{d\sigma}{dx} = \sum_{n=0}^{\infty} \alpha_s^n \left[c_n \delta(1-x) + \sum_{m=0}^{2n-1} \left(c_{nm} \left[\frac{\ln^m(1-x)}{1-x} \right]_+ + d_{nm} \ln^m(1-x) \right) + \ldots \right]$$

FACTORIZATION I: A SIMPLIFICATION FOR COMPLEX PROBLEMS

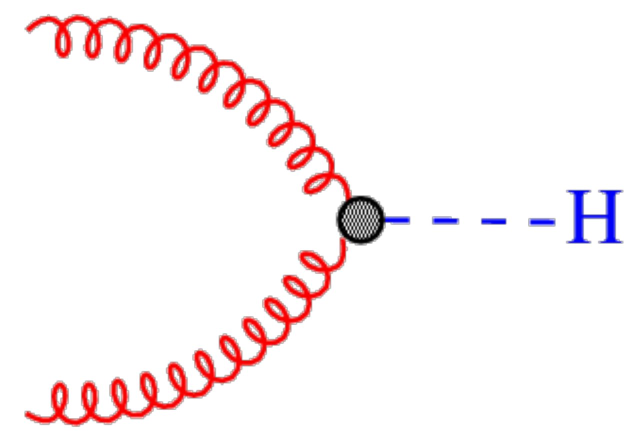
Classical example: gg → H

Top quark loop is dominated by momenta of the order of the top mass





$m_H \ll m_t$

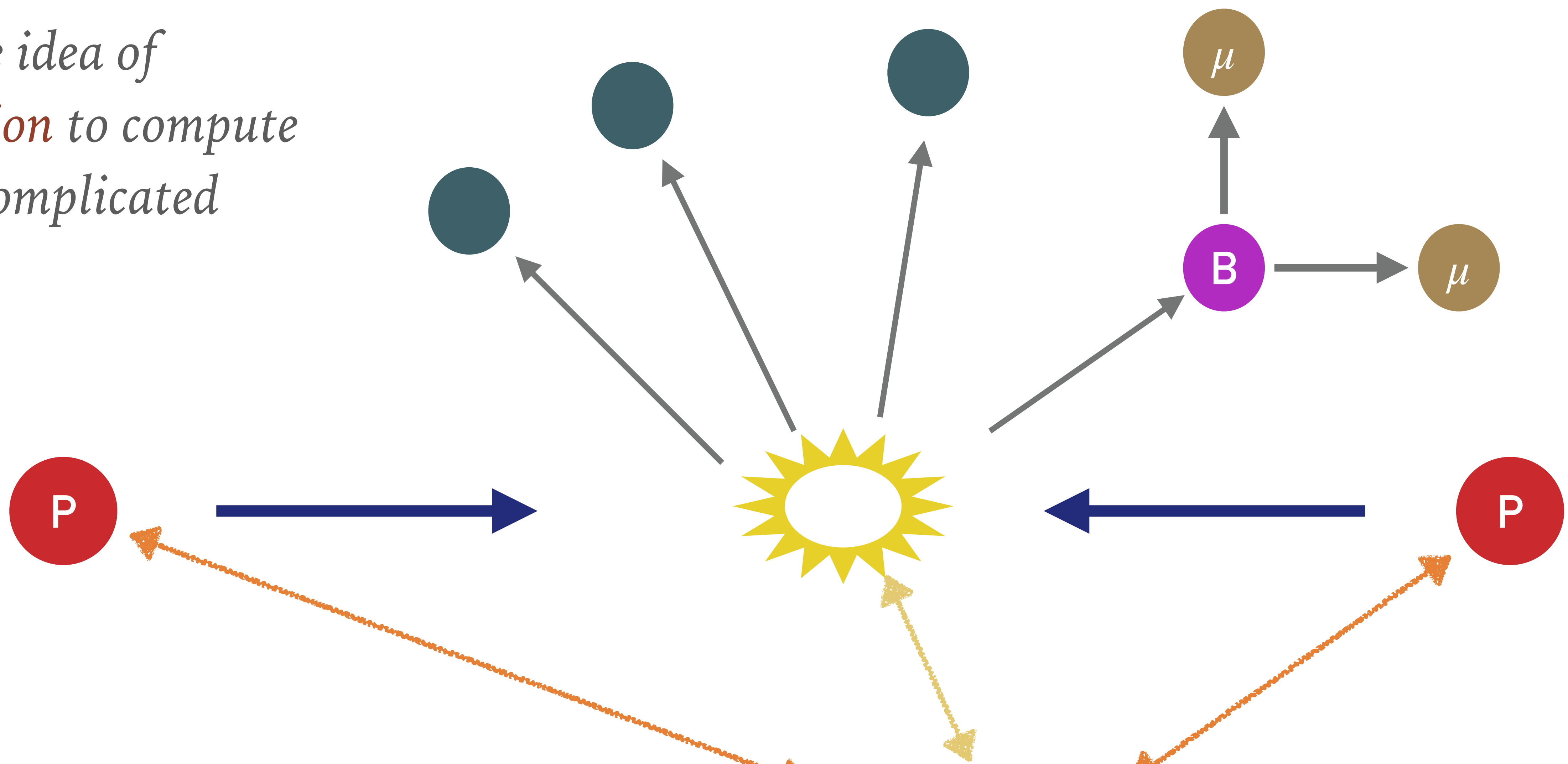


It cannot be resolved by single on-shell Higgs production so it can be absorbed into a coefficient of an effective operator

$$\mathscr{L}_{\text{eff}} = \frac{\alpha_s C_t\left(m_t, \mu\right)}{12\pi}\frac{\phi}{v}G^A_{\mu\nu}G^{\mu\nu A}$$

FACTORIZATION II: THE NECESSITY

We use the idea of factorization to compute rates for complicated processes



$$d\sigma \sim \sum_{a,b} \int_0^1 dx_a dx_b\, f_{a/A}(x_a)\, \hat{\sigma}_{ab}(z)\, f_{b/B}(x_b) + \mathcal{O}\left(\frac{\Lambda_{\mathrm{QCD}}}{Q}\right)$$

Long and short distance physics can be described separately

But now they are intertwined along the light-cone

EFT

We used precision collider physics to motivate the need for a better understanding of factorization and resummation, but this is just one of many applications

EFT & Universality

EFT allows us to define of universal objects that appear in various contexts

Typical example: PDFs and collinear factorization

Out path for today

QED corrections to muon electron conversion → Amplitude for $gg \to HH$ → Definition of threshold PDFs

PLAN FOR TODAY

- Factorization - modern EFT perspective
- Leading power example: QED corrections for muon electron conversion
- Next-to-leading power corrections:
  - Amplitude for $gg \to HH$
  - Cross-section for threshold DIS

Out path for today

QED corrections to muon electron conversion → Amplitude for $gg \to HH$ → Definition of threshold PDFs

But there are infinitely many more paths to explore …

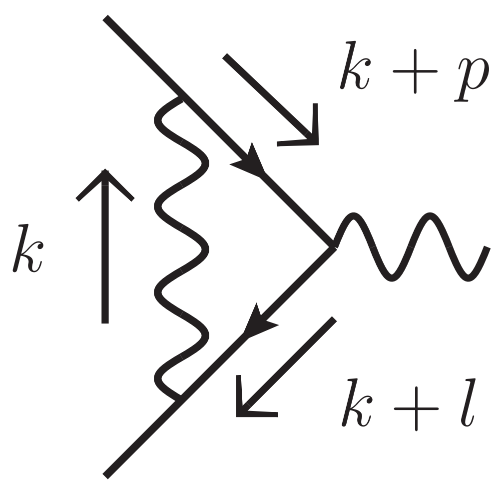
FROM THE METHOD OF REGIONS TO SCET AND FACTORIZATION

METHOD OF REGIONS

Separate via expansions the contributions to the integral from different regions, perform integrals in each region, and recover the original integral in the sum of regions. [M. Beneke, V. Smirnov, hep/9711391]

Example is the massless Sudakov form factor

Consider limit:



$$p^2 \sim l^2 \ll Q^2$$

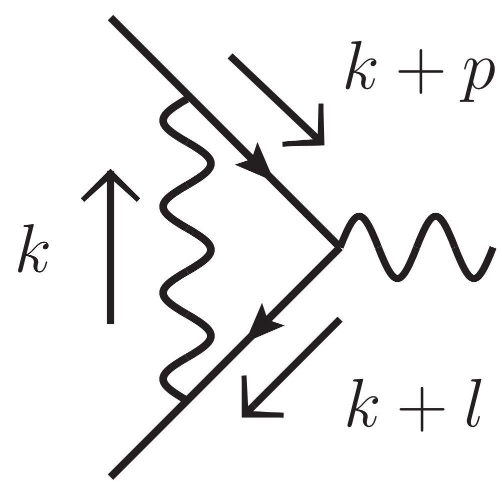
$$\frac{p^2}{Q^2} \sim \frac{l^2}{Q^2} \sim \lambda^2$$

$$I = i\pi^{-d/2}\mu^{4-d}\int d^dk \frac{1}{(k^2+i0)\,[(k+l)^2+i0]\,[(k+p)^2+i0]}$$

External momenta: $p^\mu = Q(1,\lambda^2,\lambda)$ $\quad l^\mu = Q(\lambda^2,1,\lambda)$

$$p^\mu \;=\; (n_+p)\frac{n_-^\mu}{2} + (n_-p)\frac{n_+^\mu}{2} + p_\perp^\mu \;=\; (n_+p, n_-p, p_\perp)$$

METHOD OF REGIONS



Consider limit:

$$p^2 \sim l^2 \ll Q^2$$

$$\frac{p^2}{Q^2} \sim \frac{l^2}{Q^2} \sim \lambda^2$$

Modes

$k^\mu \sim Q(1,1,1)$ $\quad k^\mu \sim Q(1,\lambda^2,\lambda)$

$k^\mu \sim Q(\lambda^2,1,\lambda)$ $\quad k^\mu \sim Q(\lambda^2,\lambda^2,\lambda^2)$

External momenta: $p^\mu = Q(1,\lambda^2,\lambda) \quad l^\mu = Q(\lambda^2,1,\lambda)$

$$p^\mu \;=\; (n_+p)\frac{n_-^\mu}{2} + (n_-p)\frac{n_+^\mu}{2} + p_\perp^\mu \;=\; (n_+p, n_-p, p_\perp)$$

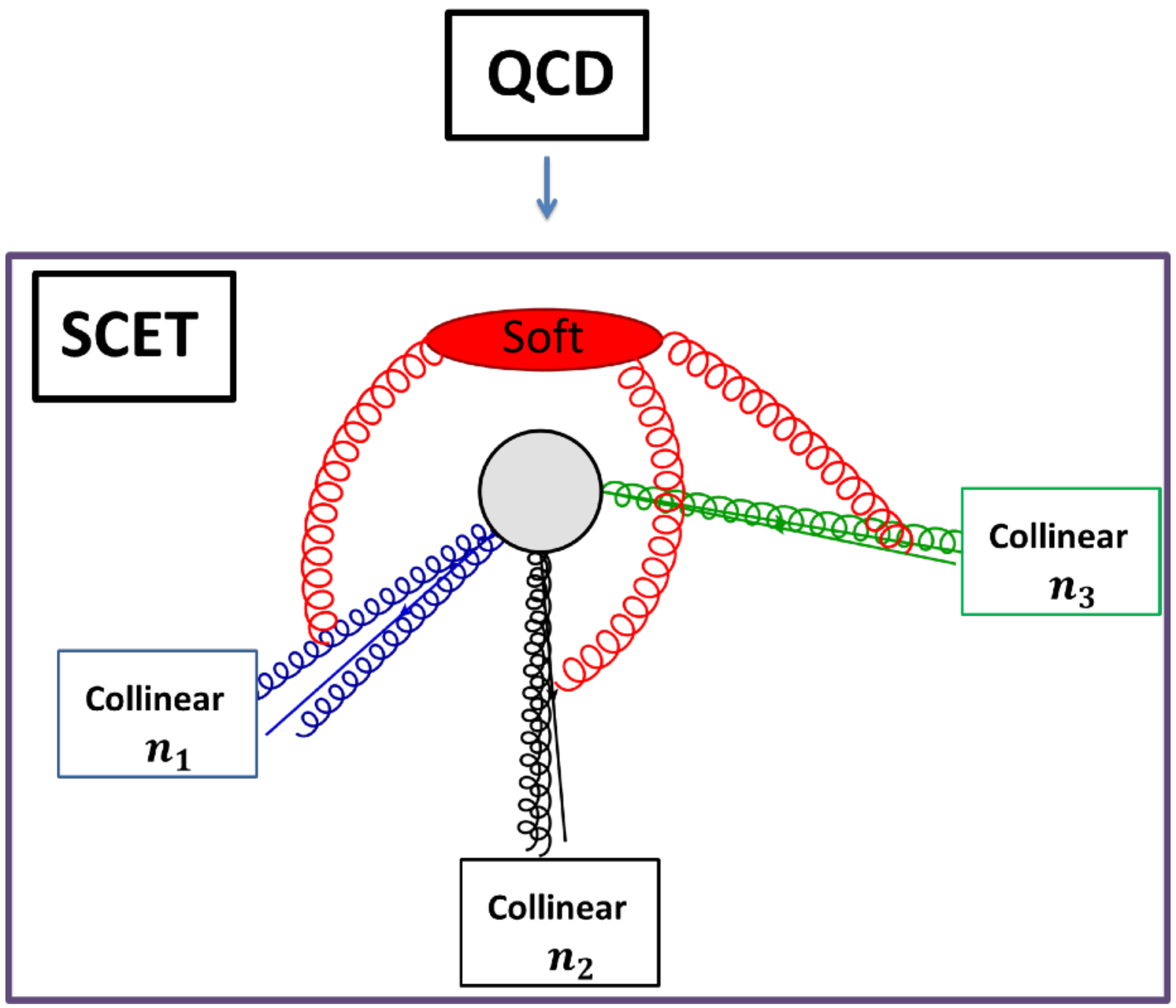
$$\begin{aligned}
I_h &= \frac{\Gamma(1+\varepsilon)}{Q^2}\left(\frac{1}{\varepsilon^2} + \frac{1}{\varepsilon}\ln\frac{\mu^2}{Q^2} + \frac{1}{2}\ln^2\frac{\mu^2}{Q^2} - \frac{\pi^2}{6} + \mathcal{O}(\lambda)\right) \\
I_c &= \frac{\Gamma(1+\varepsilon)}{Q^2}\left(-\frac{1}{\varepsilon^2} - \frac{1}{\varepsilon}\ln\frac{\mu^2}{P^2} - \frac{1}{2}\ln^2\frac{\mu^2}{P^2} + \frac{\pi^2}{6} + \mathcal{O}(\lambda)\right) \\
I_{\bar{c}} &= \frac{\Gamma(1+\varepsilon)}{Q^2}\left(-\frac{1}{\varepsilon^2} - \frac{1}{\varepsilon}\ln\frac{\mu^2}{L^2} - \frac{1}{2}\ln^2\frac{\mu^2}{L^2} + \frac{\pi^2}{6} + \mathcal{O}(\lambda)\right) \\
I_s &= \frac{\Gamma(1+\varepsilon)}{Q^2}\left(\frac{1}{\varepsilon^2} + \frac{1}{\varepsilon}\ln\frac{\mu^2 Q^2}{L^2P^2} + \frac{1}{2}\ln^2\frac{\mu^2 Q^2}{L^2P^2} + \frac{\pi^2}{6} + \mathcal{O}(\lambda)\right)
\end{aligned}$$

$$I \equiv I_h + I_c + I_{\bar{c}} + I_s = \frac{1}{Q^2}\left(\ln\frac{Q^2}{L^2}\ln\frac{Q^2}{P^2} + \frac{\pi^2}{3} + \mathcal{O}(\lambda)\right).$$

EFFECTIVE FIELD THEORY

Modes are incorporated into the Soft Collinear Effective Theory which is a low-energy EFT of QCD.

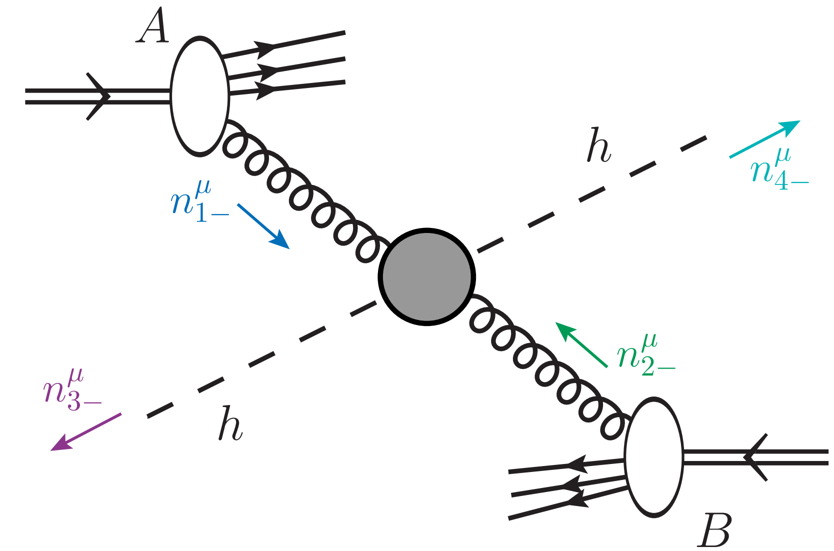
[C. Bauer, S. Fleming, D. Pirjol and I. Stewart, hep-ph/0011336] [C. Bauer, D. Pirjol, I. Stewart, hep-ph/0109045]
[M. Beneke, A. Chapovsky, M. Diehl, T. Feldmann, hep-ph/0206152] [M. Beneke, T. Feldmann, hep-ph//0211358]



- Observable specific description, with collinear sectors corresponding to large energy flow directions
- Interactions between sectors are mediated by the soft degrees of freedom
- Every interaction is well defined in terms of power counting - this allows for systematic expansion

EFFECTIVE FIELD THEORY

$$\psi(x) \to \underbrace{\psi_1(x)+\ldots+\psi_N(x)}_{N\,\text{collinear fermion fields}} + q(x) \qquad \mathscr{L}_{\text{SCET}} = \sum_{i=1}^{N} \mathscr{L}_{c_i} + \mathscr{L}_{\text{soft}}$$

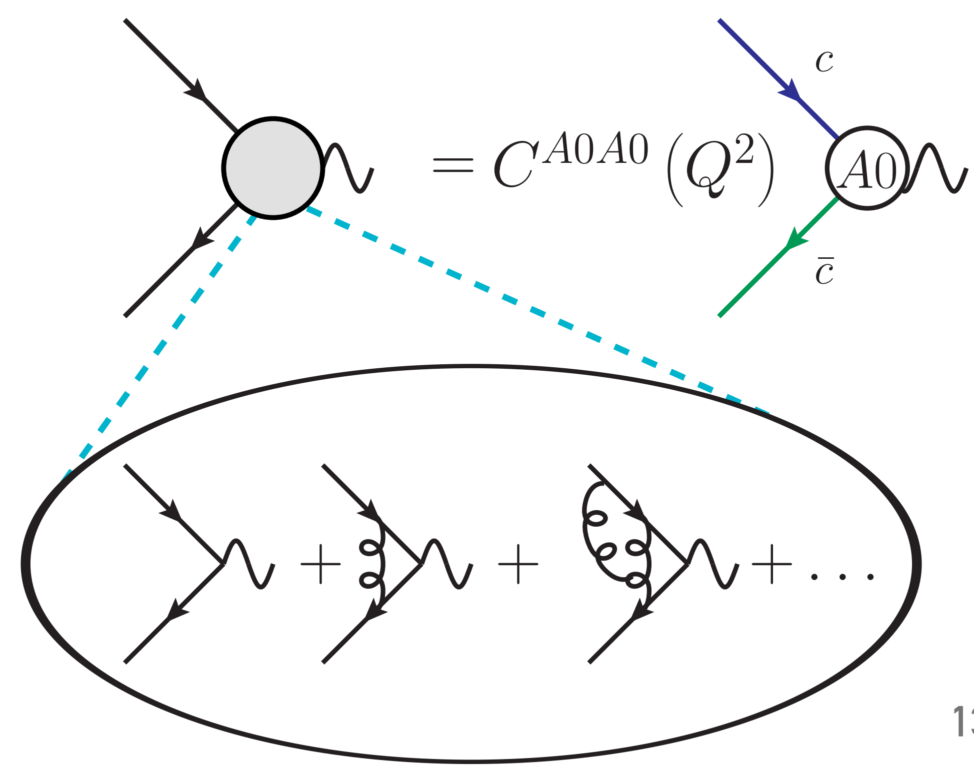


where each of the Lagrangians belonging to a collinear direction is expanded in powers of the small parameter λ

$$\mathscr{L}_{c_i} = \underbrace{\mathscr{L}^{(0)}_{c_i}}_{\text{LP}} + \underbrace{\mathscr{L}^{(1)}_{c_i}}_{\mathcal{O}(\lambda^1)} + \underbrace{\mathscr{L}^{(2)}_{c_i}}_{\mathcal{O}(\lambda^2)} + \ldots \qquad \mathscr{L}^{(1)}_c = \bar{\chi}_c\, i x^\mu_\perp \left[i n_- \partial \mathscr{B}^+_\mu \right] \frac{\not{n}_+}{2} \chi_c$$

We keep the i-collinear, and soft degrees of freedom. The hard modes are integrated out:

$$\bar{\psi}\gamma^\mu\psi = \int \prod_{i=1}^{N} \prod_{k=1}^{n_i} dt_{i_k}\, C(\{t_{i_k}\}) \prod_{i=1}^{N} J_i(t_{i_1}, \ldots, t_{i_{n_i}})$$



EFFECTIVE FIELD THEORY

We integrate-out hard scale Q

But keep particles with energies~Q

The EFT becomes non-local

Factorization is not multiplicative like in HTEFT and convolutions appear

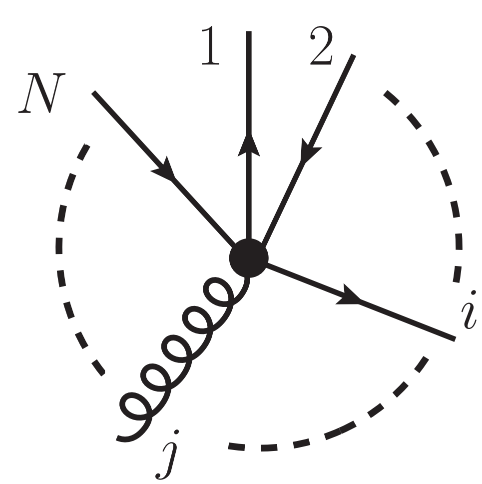
Generic N-jet operator has the form:

[M. Beneke, M. Garny, R. Szafron, J. Wang, 1712.04416, 1712.07462, 1808.04742, 1907.05463]

$$J = \int \prod_{i=1}^{N} \prod_{k=1}^{n_i} dt_{i_k}\, C(\{t_{i_k}\}) \prod_{i=1}^{N} J_i(t_{i_1}, t_{i_2}, \dots, t_{i_{n_i}})$$

where the J_i are constructed using collinear gauge invariant building blocks in the same direction

$$\chi_i(t_i n_{i+}) \equiv W_i^\dagger \xi_i \qquad \mathcal{A}^{\mu}_{i\perp}(t_i n_{i+}) \equiv W_i^\dagger [iD^{\mu}_{\perp i} W_i]$$



$$W_{c_i}(x) = \mathbf{P}\exp\left[ig_s \int_{-\infty}^{0} ds\, n_{i+} \cdot A_{c_i}(x + s n_{i+})\right]$$

LEADING POWER EXAMPLE:

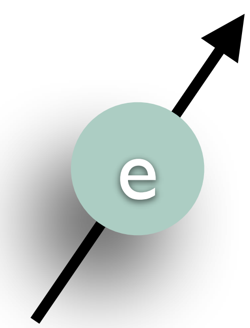
QED CORRECTIONS FOR MUON-ELECTRON CONVERSION

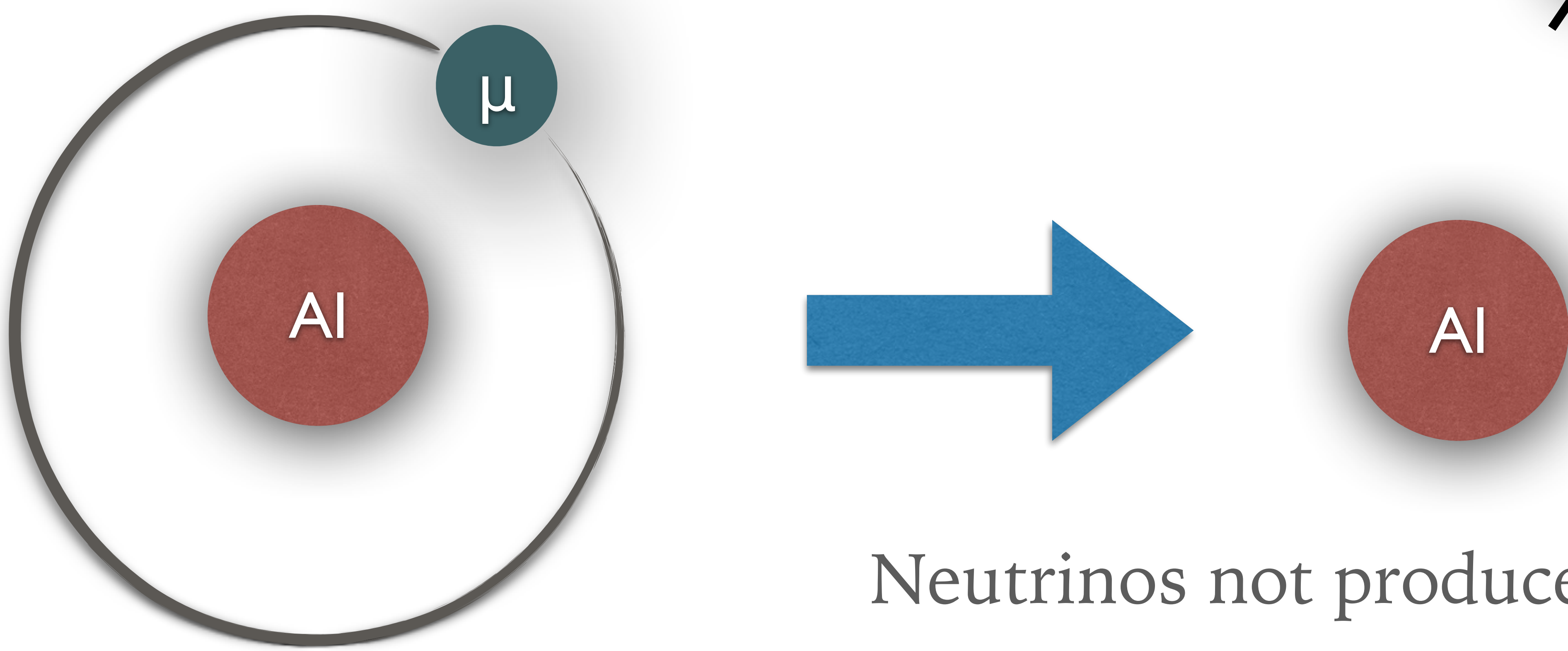
[2412.05702 with Duarte Fontes]

MUON ELECTRON COHERENT CONVERSION

$$-\frac{4G_F}{\sqrt{2}}\sum_{X=L,R}\Big\{C_{SX}\,\bar{e}P_X\mu\,\bar{N}N + C_{PX}\bar{e}P_X\mu\,\bar{N}\gamma_5 N + C_{VX}\,\bar{e}\gamma^\alpha P_X\mu\,\bar{N}\gamma_\alpha N$$
$$+ C_{AX}\,\bar{e}\gamma^\alpha P_X\mu\,\bar{N}\gamma_\alpha\gamma_5 N + C_{\mathrm{Der}X}\,\bar{e}\gamma^\alpha P_X\mu\,(\bar{N}\overleftrightarrow{\partial}_\alpha i\gamma_5 N)$$
$$+ C_{TX}\,\bar{e}\sigma^{\alpha\beta}P_X\mu\,\bar{N}\sigma_{\alpha\beta}N\Big\} + \text{h.c.}\,,$$

$E_e \approx m_\mu$





Neutrinos not produced

CLFV

MUON ELECTRON CONVERSION

$$E_{\text{max}} = m_\mu + E_b + E_{\text{rec}}$$

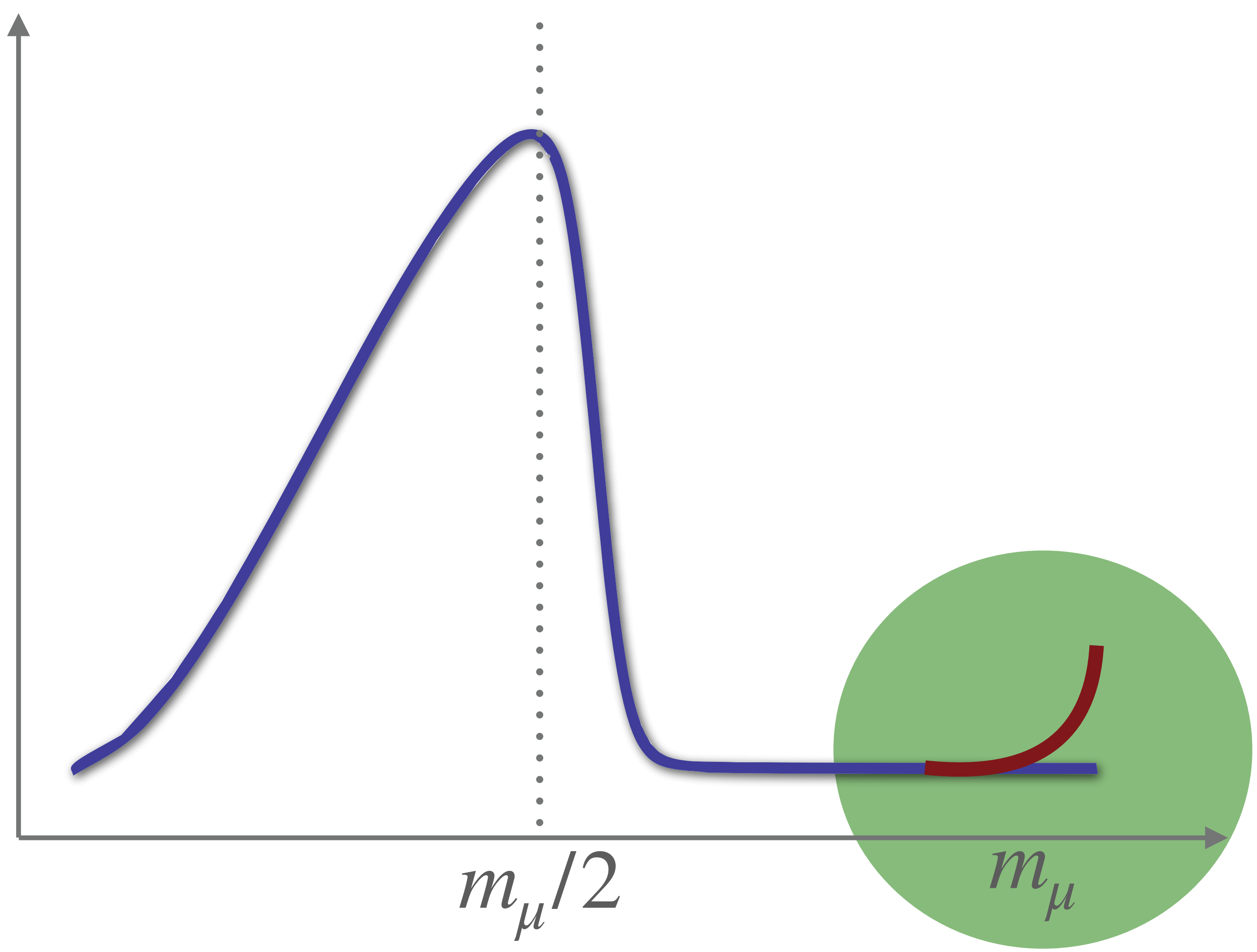
$$E_b \approx -m_\mu \frac{(Z\alpha)^2}{2}$$

Binding energy

$$E_{rec} \approx -\frac{m_\mu^2}{2m_N}$$

Recoil energy

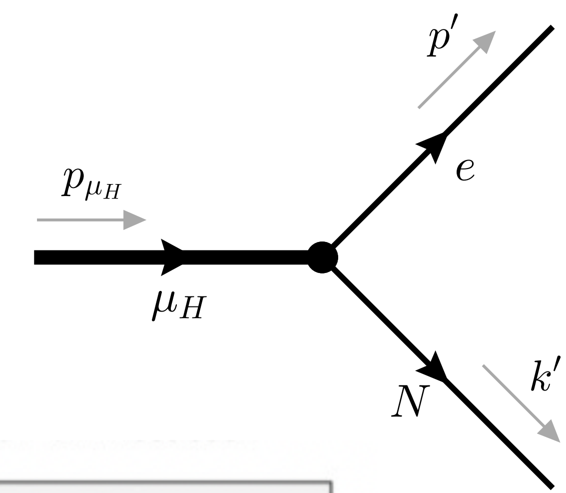
(kinetic energy of the nucleus)



Conversion Signal

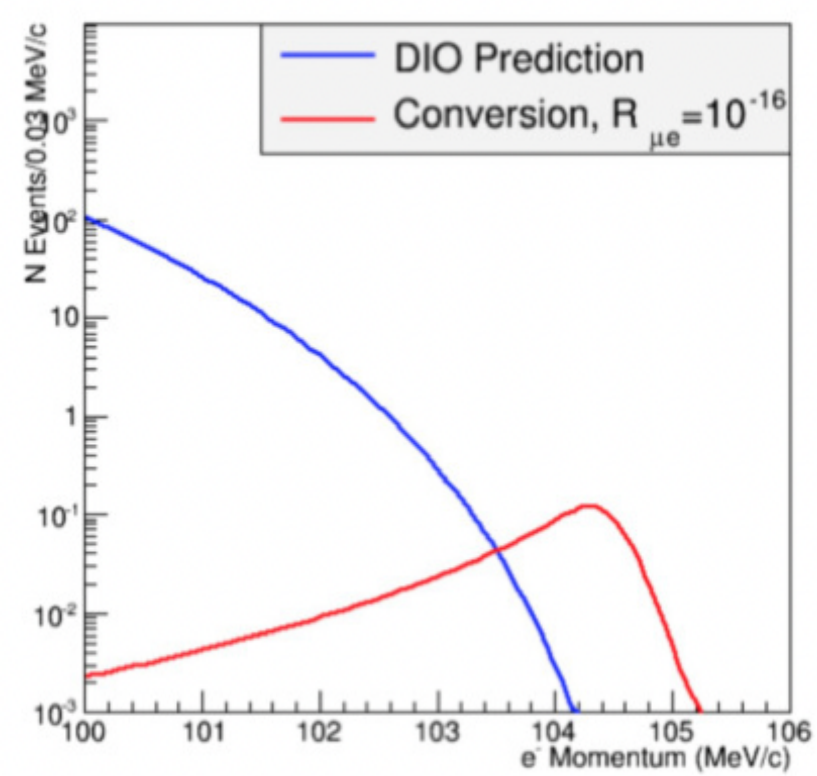
QED CORRECTIONS

At LO, the conversion is a 2-body decay



QED effects generate non-trivial spectrum

Crucial for experiments to optimize signal/background ratio



QED effects also change the overall normalization

Important for extraction of bounds on New Physics models

SCALES IN THE PROBLEM

hard scale: $\mu_h \sim m_\mu \simeq E_e$

Electron energy is large: SCET

semi-hard scale: $\mu_{sh} \sim Z\alpha m_\mu$

Bound state muon: NRQED

Bound muon velocity

Massive energetic particle: bHQET

soft scale: $\mu_s \sim (Z\alpha)^2 m_\mu \simeq m_e \simeq \Delta E$

Bound muon energy, electron mass and energy resolution

soft-collinear scale: $\mu_{sc} \sim m_e \dfrac{\Delta E}{m_\mu}$

$\lambda \sim m_e/m_\mu$ *Induced soft-collider scale below electron mass*

$$p' = m_e v_e + \hat{p}' \qquad \hat{p}' \sim m_\mu \lambda^2 \left(1, \lambda^2, \lambda^4\right)$$

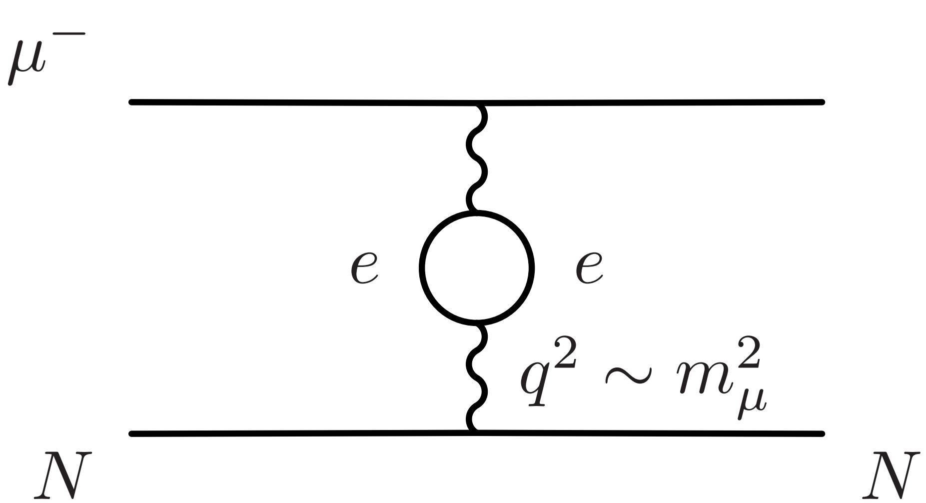
FACTORIZATION

$$\frac{1}{\Gamma_{\mathrm{LO}}}\frac{d\Gamma_{\mu_H\to eN\mathscr{X}}}{dE_e} = |\psi_{\mathrm{corr}}|^2 |C_{\mathrm{hard}}|^2 |C_m(m_e)|^2 \int dE_{sc}\, dE_s\, \delta(\Delta E - E_{sc} - E_s)\mathscr{S}(E_s)\mathscr{S}\mathscr{C}(E_{sc})$$

$$|\psi_{\mathrm{corr}}|^2 \equiv \frac{|\psi_{\mathrm{Schr.}}(0)|^2}{|\psi_{\mathrm{Schr.}}(0)|^2_{\mathrm{LO}}}$$

Corrections to the wave-function

The dominant one comes from the long-range VP potential



$$r \sim \frac{1}{m_e} \gg \frac{1}{m_\mu Z\alpha}$$

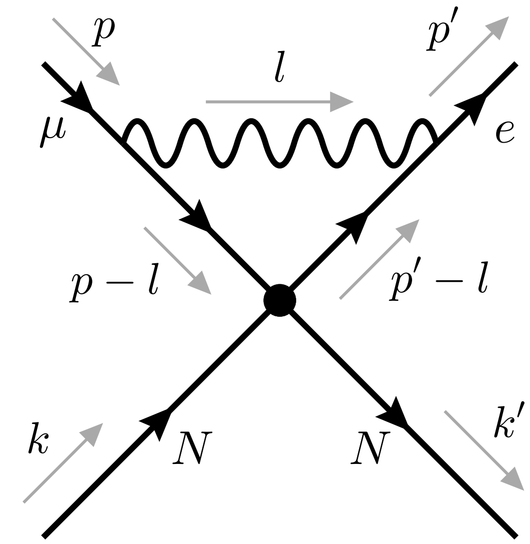
VP range — Muons orbit radius

Proper EFT definition allows to systematically include higher order effects

FACTORIZATION

$$\frac{1}{\Gamma_{\text{LO}}}\frac{d\Gamma_{\mu_H\to eN\mathscr{X}}}{dE_e} = |\psi_{\text{corr}}|^2 |C_{\text{hard}}|^2 |C_m(m_e)|^2 \int dE_{sc}\, dE_s\, \delta(\Delta E - E_{sc} - E_s)\mathscr{S}(E_s)\mathscr{S}\mathscr{C}(E_{sc})$$

Hard matching corrections



$$C_{\text{hard}} = 1 - \frac{\alpha}{4\pi}\left\{\frac{m_\mu \ln\left(\frac{16E_e^4\mu^2}{m_\mu^6}\right)}{4E_e - 2m_\mu} - \ln\left(\frac{\mu}{m_\mu}\right)\ln\left(\frac{\mu^7}{4E_e^2 m_\mu^5}\right) - 2\mathrm{Li}_2\left(1 - \frac{m_\mu}{2E_e}\right) + \ln^2\left(\frac{2E_e m_\mu^2}{\mu^3}\right) + \frac{2E_e \ln\left(\frac{m_\mu}{\mu}\right)}{2E_e - m_\mu} + \frac{\pi^2}{12}\right\} + \mathscr{O}(\alpha^2)$$

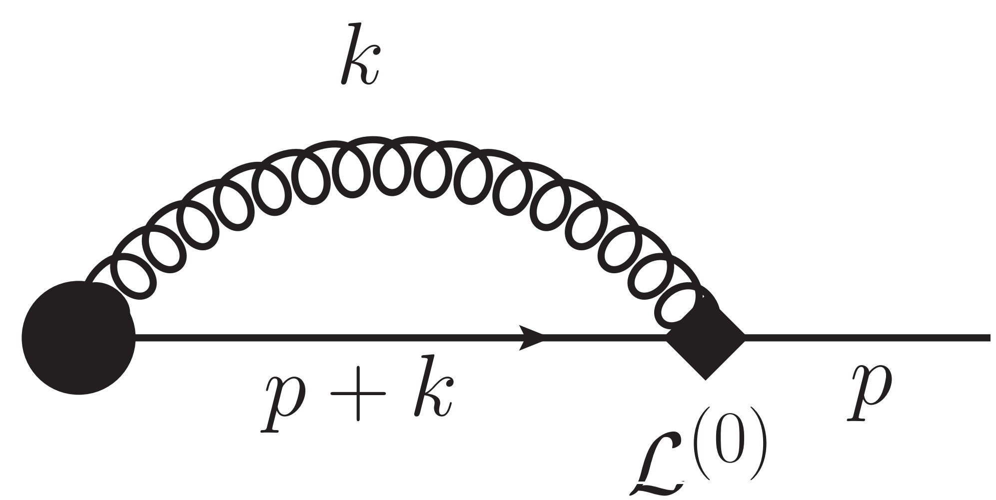
EFT allows to systematically separate various effects and avoid double-counting

FACTORIZATION

$$\frac{1}{\Gamma_{\text{LO}}}\frac{d\Gamma_{\mu_H\to eN\mathcal{X}}}{dE_e} = |\psi_{\text{corr}}|^2 |C_{\text{hard}}|^2 |C_m(m_e)|^2 \int dE_{sc}\, dE_s\, \delta(\Delta E - E_{sc} - E_s)\mathcal{S}(E_s)\mathcal{S}\mathscr{C}(E_{sc})$$

Radiative jet function

or matching from massive SCET to bHQET



$$C_m(m_e;\mu) = 1 + \frac{\alpha}{4\pi}\left\{2\ln^2\left(\frac{m_e}{\mu}\right) - \ln\left(\frac{m_e}{\mu}\right) + \frac{\pi^2}{12} + 2\right\}$$

We are “freezing” the large momentum component of the electron so the collinear splitting are not allowed below the m_e scale

NOTE ON THE REAL RADIATION

$$\frac{1}{\Gamma_{\rm LO}}\frac{d\Gamma_{\mu_H\to eN\mathcal{X}}}{dE_e} = |\psi_{\rm corr}|^2|C_{\rm hard}|^2|C_m(m_e)|^2\int dE_{sc}\,dE_s\,\delta(\Delta E - E_{sc} - E_s)\mathcal{S}(E_s)\mathcal{S}\mathcal{C}(E_{sc})$$

Convolution of the soft and soft—collinear functions

$$\int dE_{sc}\,dE_s\,\delta(\Delta E - E_{sc} - E_s)\mathcal{S}(E_s)\mathcal{S}\mathcal{C}(E_{sc})$$

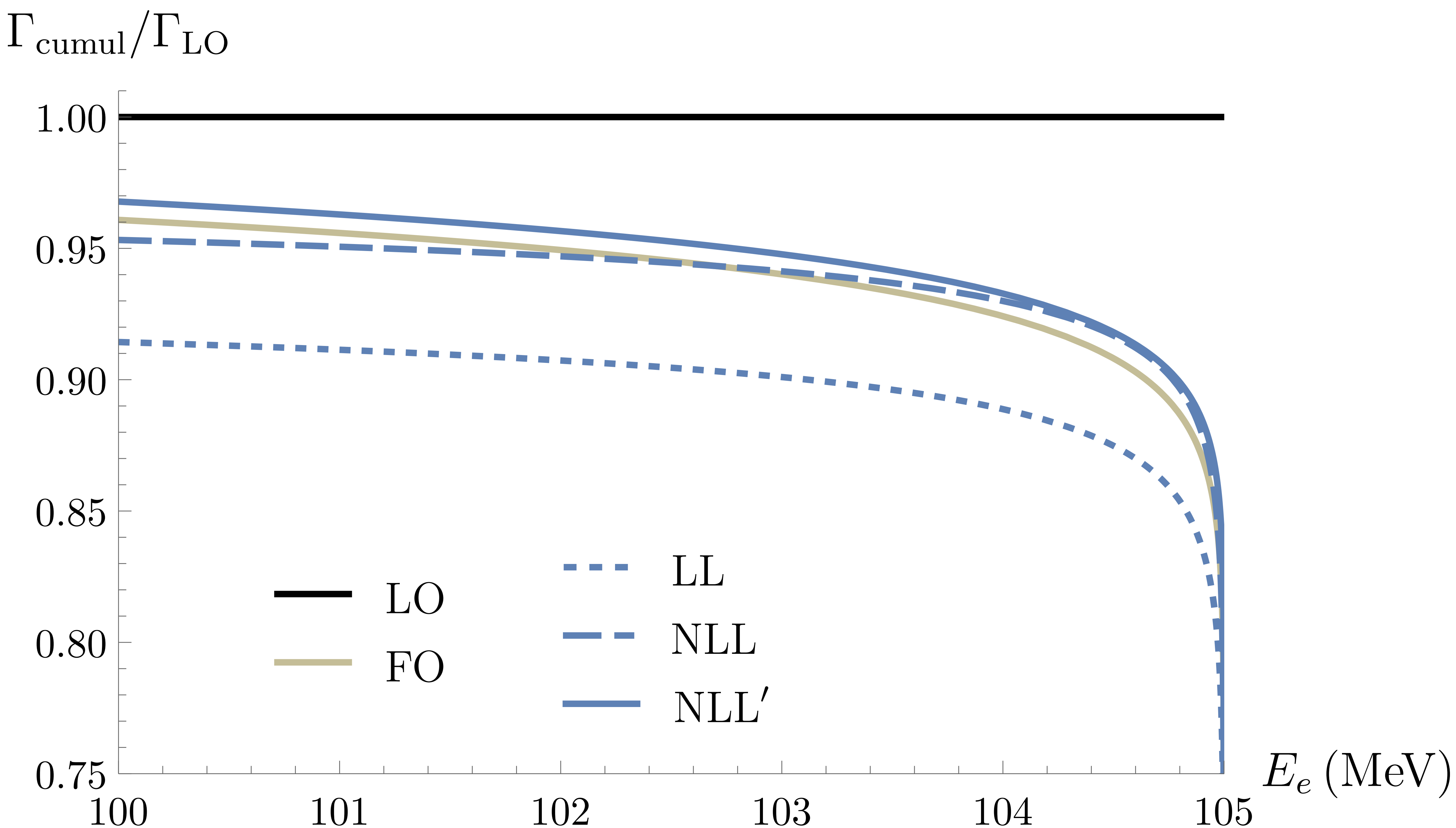
is equivalent to YFS QED soft function

$$\delta_{\rm real} \sim \frac{\alpha}{\pi}\left\{\ln\left(\frac{m_\mu \mu^2}{2\Delta E^2 m_e}\right) + \ln\left(\frac{2m_\mu}{m_e}\right)\ln\left(\frac{2\Delta E^2 m_e}{m_\mu \mu^2}\right) - \frac{\pi^2}{6} + 1\right\}$$

$$\mathcal{S}(E_s) = \int \frac{dt}{2\pi} e^{itE_s}\langle 0|\mathcal{O}_s(t)\mathcal{O}_s(0)\big|0\rangle \qquad \mathcal{O}_s(x) = \left[Y_v^{s\dagger}\overline{Y}_v^s Y_{v_e}^{s\dagger}\overline{Y}_v^s\right](x)$$

$$\mathcal{S}\mathcal{C}(E_{sc}) = \int \frac{dt}{2\pi} e^{itE_{sc}}\langle 0|\mathcal{O}_{sc}(t)\mathcal{O}_{sc}(0)\big|0\rangle \qquad \mathcal{O}_{sc}(x) = \left[Y_v^{sc\dagger}\overline{Y}_v^{sc} Y_{v_e}^{sc\dagger}\overline{Y}_v^{sc}\right](x)$$

NUMERICAL IMPACT OF QED CORRECTIONS



NEXT-TO-LEADING POWER EXAMPLE:

AMPLITUDE FOR $gg \to HH$

[2501.00587 with Sebastian Jaskiewicz, Stephen Jones, and Yannick Ulrich]

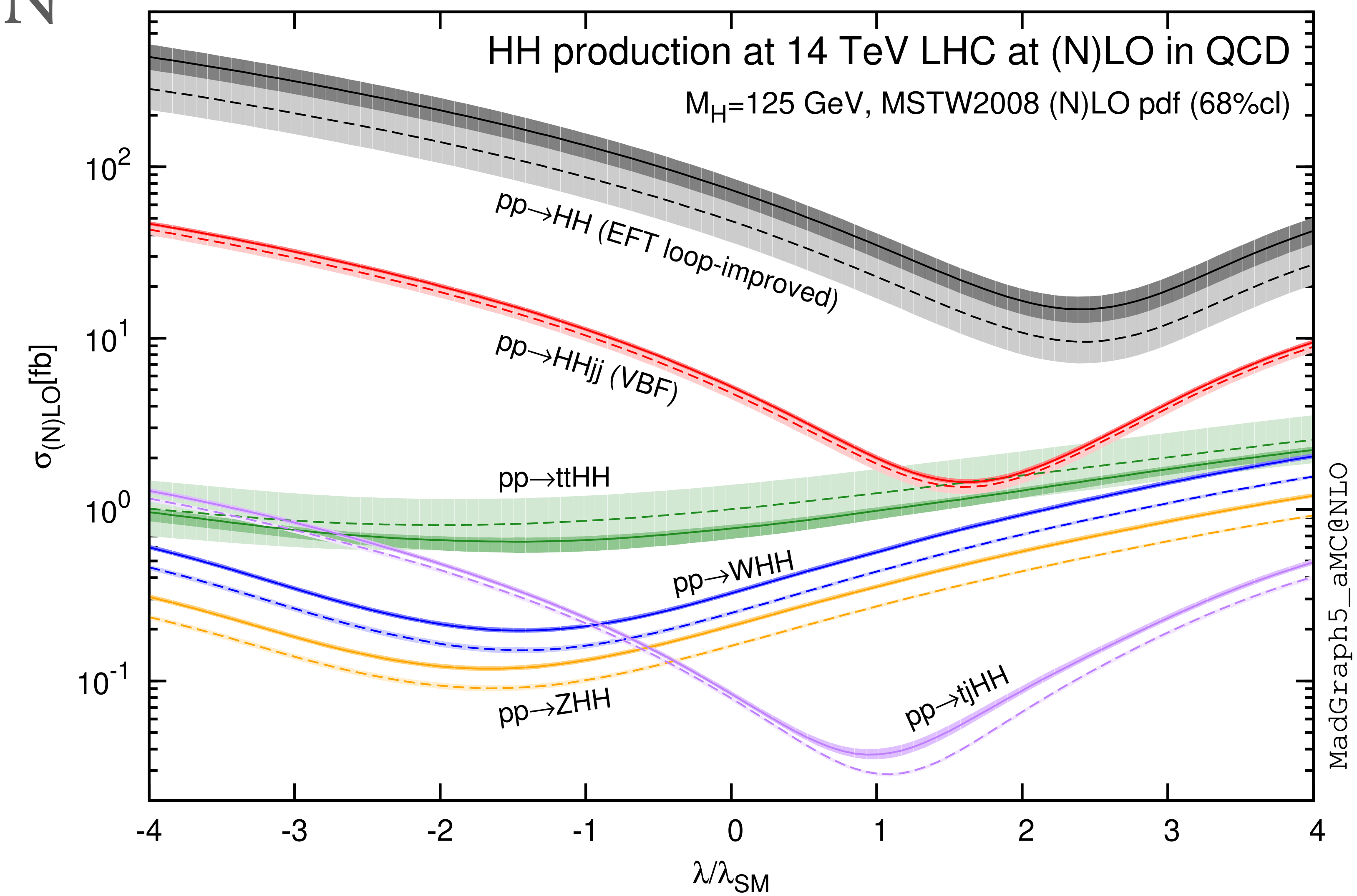
SCALE UNCERTAINTY

Combination of NLO and N^m LO HTL yields:

- Scale uncertainty of: + 2 % / -5 %

- PDF $+\alpha_s$: ± 3.0 %

- m_T approx: ± 2.7 %



[Chen, Li, Shao, Wang 19, 19]
[Grazzini, Heinrich, Jones, Kallweit, Kerner, Lindert, Mazzitelli 18;]
[de Florian, Grazzini, Hanga, Kallweit, Lindert, Maierhöfer, Mazzitelli, Rathlev 16]
[Maltoni, Vryonidou, Zaro 14 (recalculated);]
[Dawson, Dittmaier, Spira 98 (recalculated); Glover, van der Bij 88 (recalculated)]
[Borowka, Greiner, Heinrich, Jones, Kerner, Schlenk, Schubert, Zirke 16]

[R. Frederix, S. Frixione, V. Hirschi, F. Maltoni, O. Mattelaer, P. Torrielli, E. Vryonidou, and M. Zaro, 1401

[J. Baglio, A. Djouadi, R. Gröber, M. M. Mühlleitner, J. Quevillon, and M. Spira, 1212.5

MASS SCHEME UNCERTAINTIES

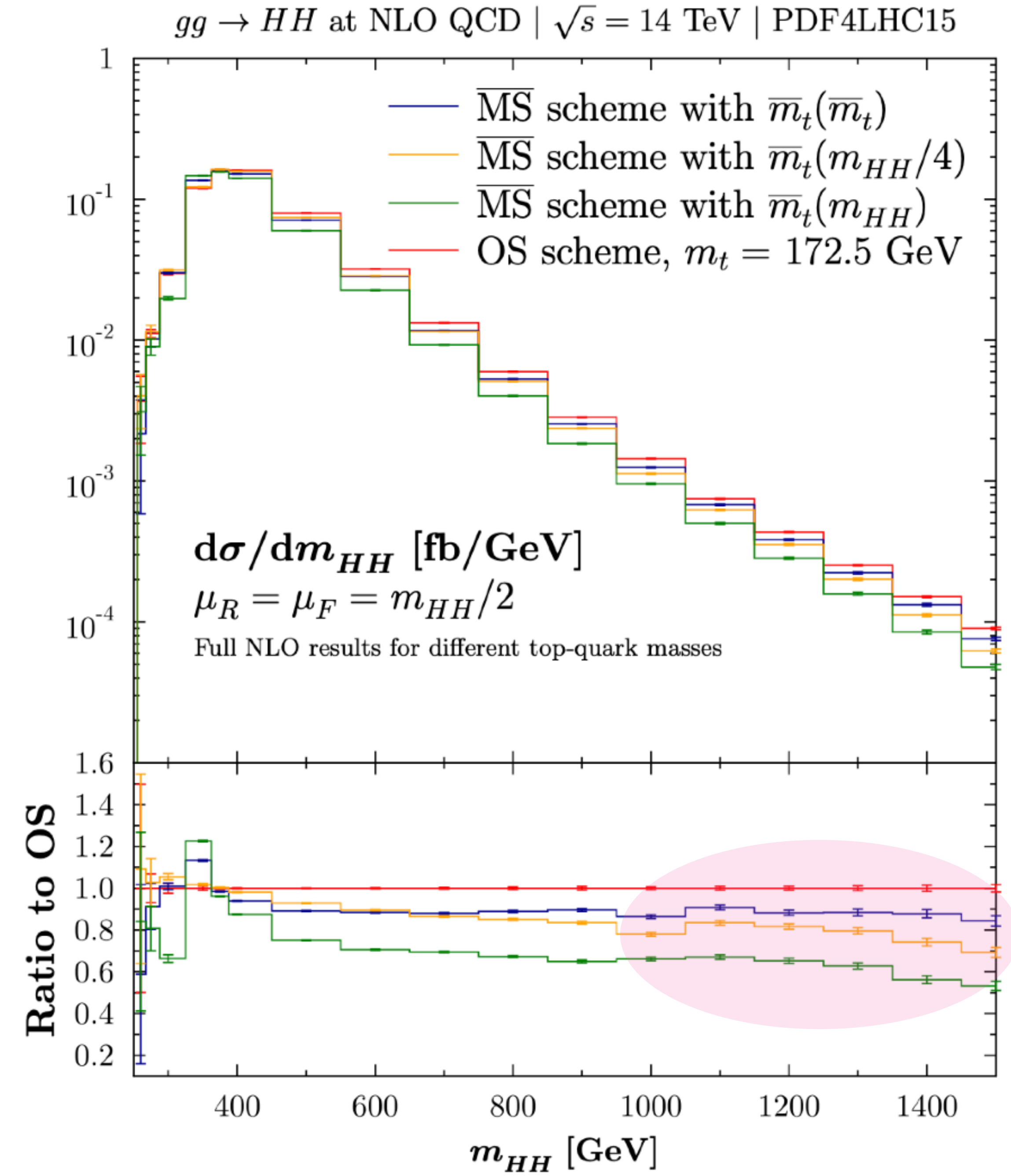
Converting the top quark mass to the $\overline{\text{MS}}$ scheme

$$m_t \to \overline{m}_t(\mu_t)\left(1 + \frac{\alpha_s(\mu_R)}{4\pi} C_F \left\{4 + 3\log\left[\frac{\mu_t^2}{\overline{m}_t(\mu_t)^2}\right]\right\}\right)$$

This leads to an addition uncertainty related to the choice of the top-quark mass scheme

$$\sqrt{s} = 13\ \text{TeV}: \quad \sigma_{tot} = 27.73(7)^{+4\%}_{-18\%}\ \text{fb},$$
$$\sqrt{s} = 14\ \text{TeV}: \quad \sigma_{tot} = 32.81(7)^{+4\%}_{-18\%}\ \text{fb},$$

[J. Baglio, F. Campanario, S. Glaus, M. Muehlleitner, J. Ronca, M. Spira, J. Streicher, 2003.03227, 2008.11626]

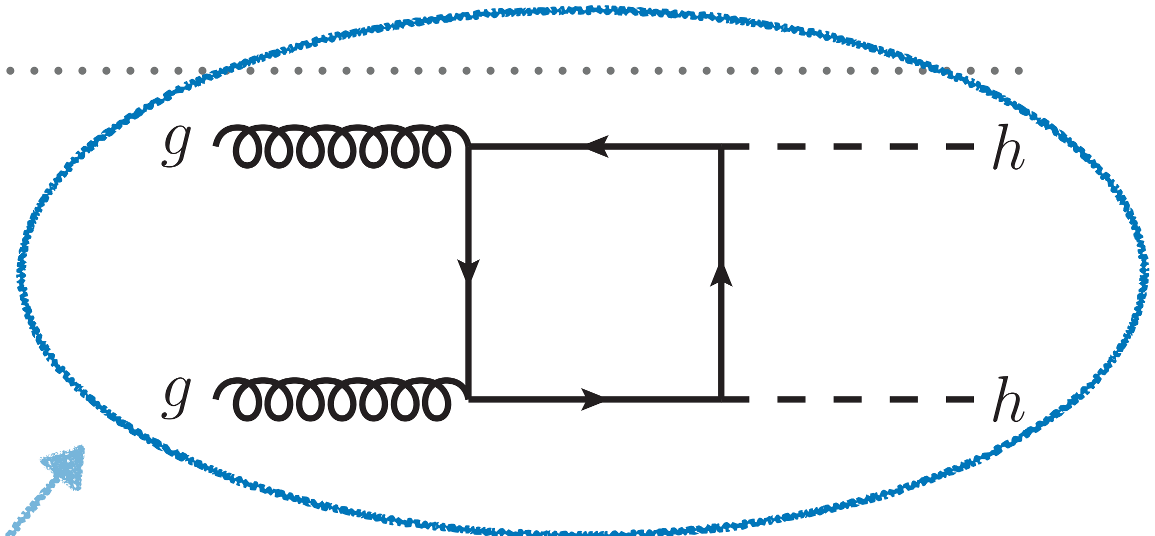


Large uncertainties in the high-energy limit

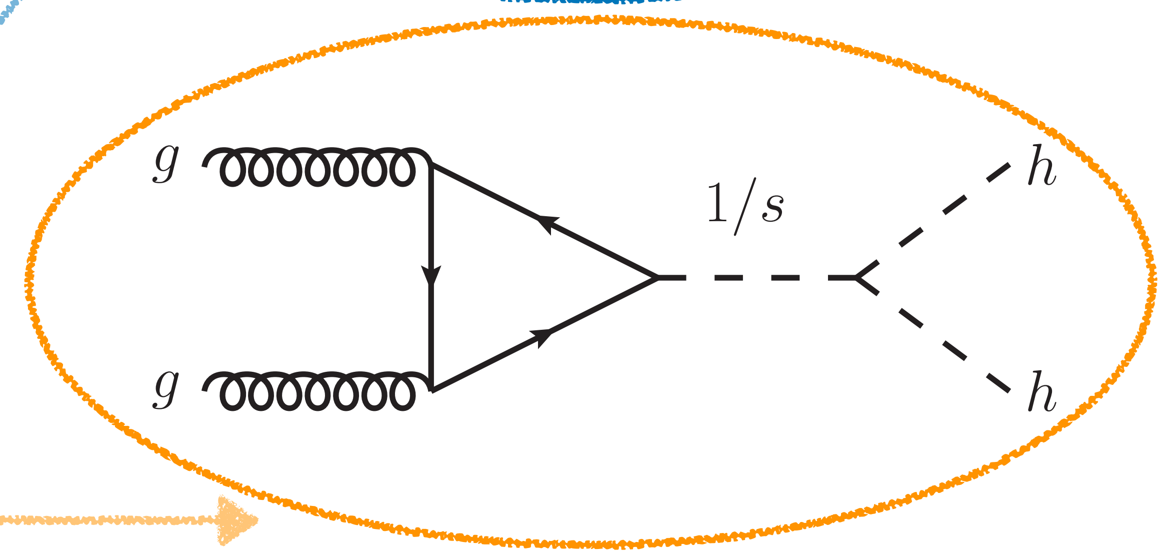
$gg \to hh$ AMPLITUDES

Structure of QCD corrections in the amplitude

$$\mathcal{M} = \varepsilon_{1,\mu}\,\varepsilon_{2,\nu}\,\delta^{AB}\left(A_1 P_1^{\mu\nu} + A_2 P_2^{\mu\nu}\right)$$



$$A_1 = T_F \frac{G_F}{\sqrt{2}}\frac{\alpha_s}{2\pi} s \left[\frac{3m_H^2}{s - m_H^2} A_{1,y_t\lambda} + A_{1,y_t^2}\right]$$



$$A_2 = T_F \frac{G_F}{\sqrt{2}}\frac{\alpha_s}{2\pi} s \left[A_{2,y_t^2}\right]$$

$$P_1^{\mu\nu} = g^{\mu\nu} - \frac{n_{1-}^{\nu} n_{2-}^{\mu}}{n_{1-}\cdot n_{2-}}$$

$$P_2^{\mu\nu} = g^{\mu\nu} + \frac{n_{1-}\cdot n_{2-}\; n_{3-}^{\nu} n_{3-}^{\mu}}{n_{1-}\cdot n_{3-}\; n_{2-}\cdot n_{3-}} - \frac{n_{1-}^{\nu} n_{3-}^{\mu}}{n_{1-}\cdot n_{3-}} - \frac{n_{3-}^{\nu} n_{2-}^{\mu}}{n_{3-}\cdot n_{2-}}$$

HIGH-ENERGY LIMIT

$$s, |t|, |u| \gg m_t^2 \gg m_H^2$$

$gg \to hh$

[J. Davies, G. Mishima, M. Steinhauser, D. Wellmann, 1811.05489]

[J. Baglio, F. Campanario, S. Glaus, M. Muehlleitner, J. Ronca, M. Spira, J. Streicher, 2008.11626]

$$A_{i,y_t^2}^{(0)} = y_t^2\, f_i(s,t) + y_t^2\, \mathcal{O}(m_t^2)$$

$$A_{i,y_t^2}^{(1)} = 6C_F\, A_i^{(0)} \log\left[\frac{m_t^2}{s}\right] + y_t^2\, \mathcal{O}(m_t^2)$$

For $gg \to hh$, the leading logarithm in m_t^2 comes from the mass counter term. In $\overline{\mathrm{MS}}$, this is converted to $\log[\mu_t^2/s]$ $\to$ choose scale $\mu_t^2 \sim s$

$gg \to Zh$

[J. Davies, G. Mishima, M. Steinhauser, 20]

[Chen, Davies, Heinrich, Jones, Kerner, Mishima, Schlenk, Steinhauser, 22]

$$A_i^{(0)} = y_t^2\, f_i(s,t)\log^2\left[\frac{m_t^2}{s}\right] + y_t^2\, \mathcal{O}(m_t^2)$$

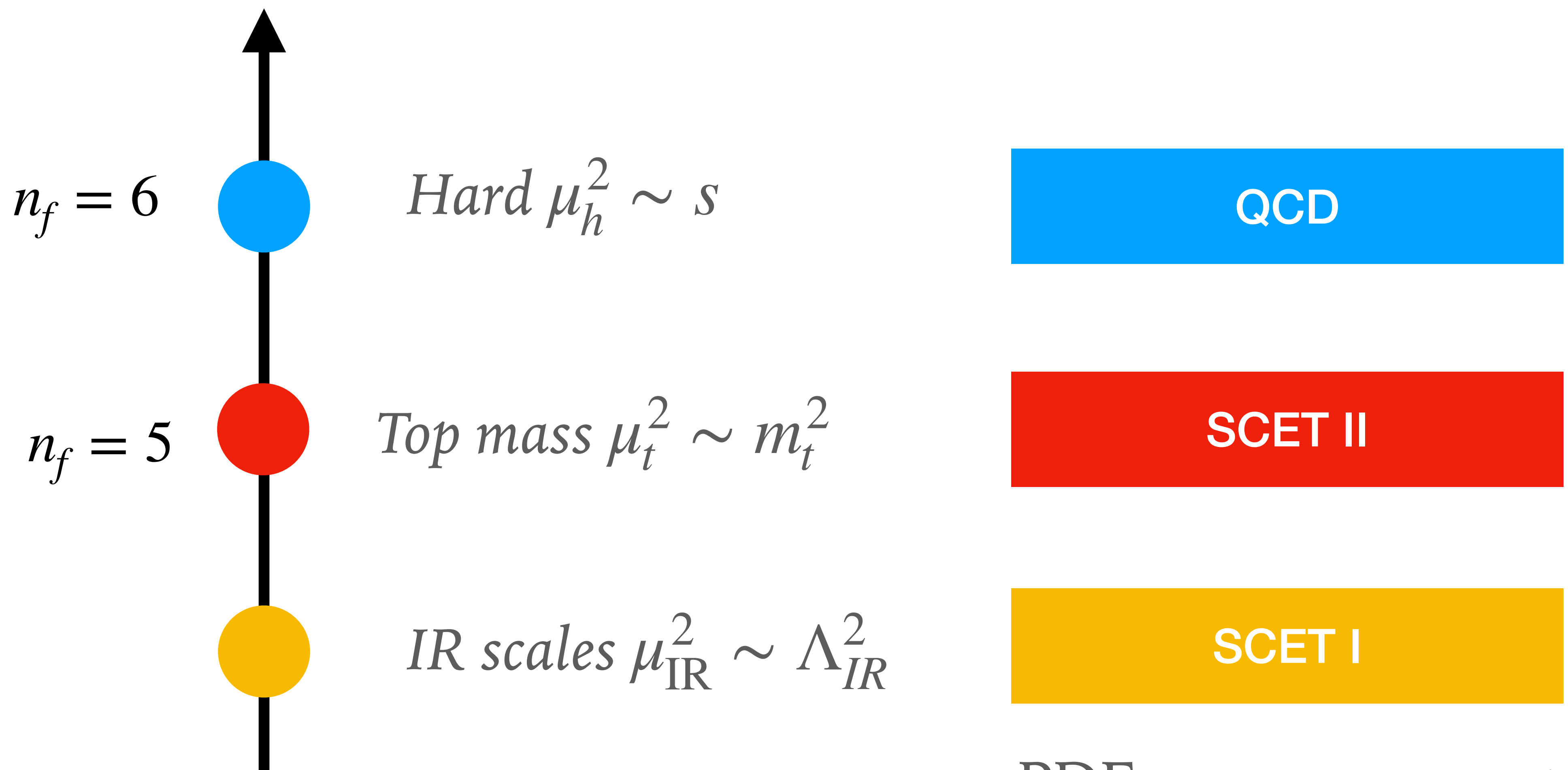
$$A_i^{(1)} = \frac{(C_F - C_A)}{6}\, A_i^{(0)} \log^2\left[\frac{m_t^2}{s}\right] + y_t^2\, \mathcal{O}(m_t^2)$$

Similar power suppressed mass logarithms studied for single Higgs

[Liu, Penin 18; Anastasiou, Penin 20; Liu, Modi, Penin 22; Liu, Neubert, Schnubel, Wang 22]

SCALES AND EFTS

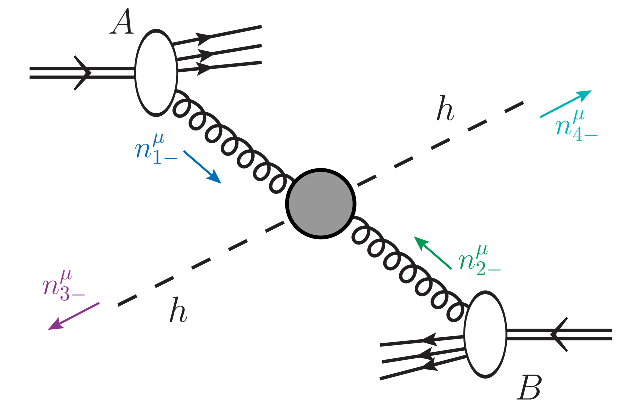
Our problem has three scales $s, |t|, |u| \gg m_t^2 \gg \Lambda_{\mathrm{IR}}^2$



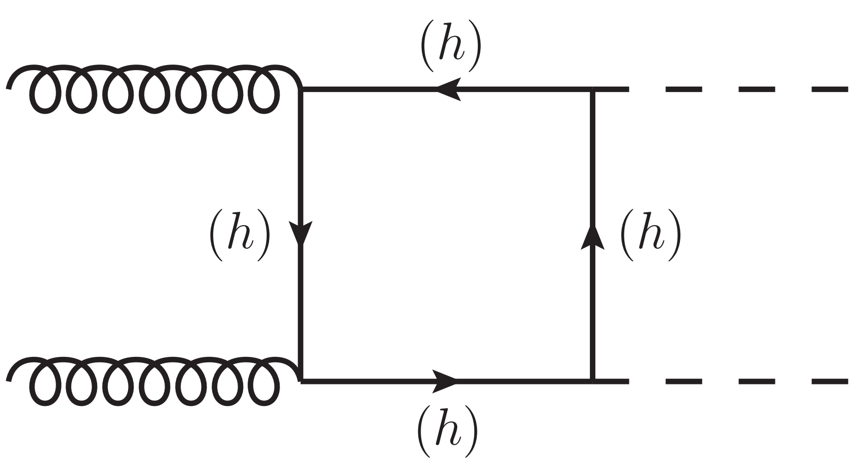
PDFs; measurement-induced scales

HARD SCALE $\mu^2 \sim s$

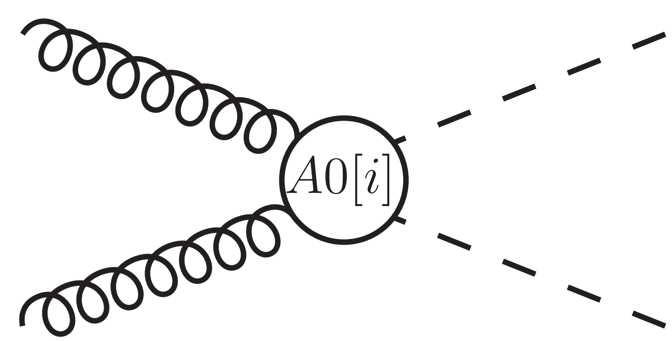
Leading power: only a single operator family allowed



$$J^{[i]}_{\mathrm{LP}}(t_1, t_2, t_3, t_4) = y_t^2\, P_i^{\mu\nu}\, \mathscr{A}_{c_1\perp_1\,\mu}(t_1 n_{1+})\, \mathscr{A}_{c_2\perp_2\,\nu}(t_2 n_{2+})\, h_{c_3}(t_3 n_{3+})\, h_{c_4}(t_4 n_{4+})$$



$$+ \dots = C^{A0}_{[i]}(s,t)$$



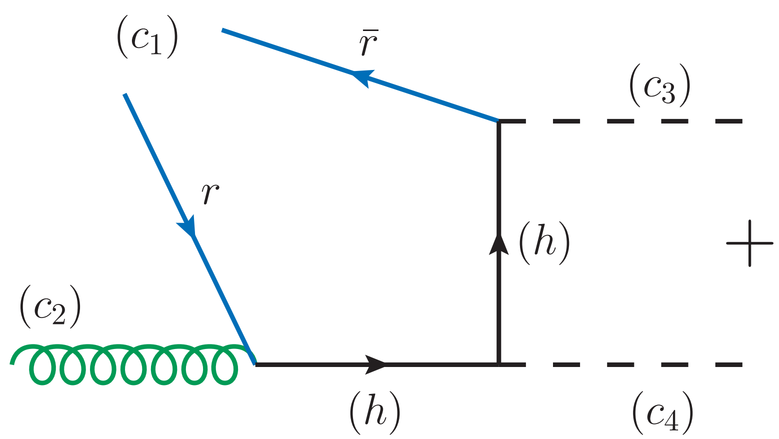
Contribution starts at one loop

$$P_1^{\mu\nu} = g^{\mu\nu} - \frac{n^\nu_{1-} n^\mu_{2-}}{n_{1-}\cdot n_{2-}} \qquad P_2^{\mu\nu} = g^{\mu\nu} + \frac{n_{1-}\cdot n_{2-}\; n^\nu_{3-} n^\mu_{3-}}{n_{1-}\cdot n_{3-}\; n_{2-}\cdot n_{3-}} - \frac{n^\nu_{1-} n^\mu_{3-}}{n_{1-}\cdot n_{3-}} - \frac{n^\nu_{3-} n^\mu_{2-}}{n_{3-}\cdot n_{2-}}$$

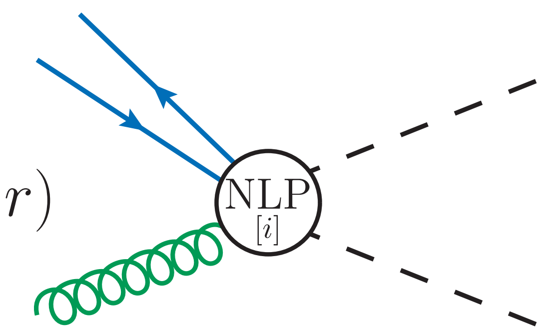
HARD SCALE $\mu^2 \sim s$

First sub-leading power: multiple hard operators allowed

$$\mathcal{M}^{\mathrm{QCD}}_{\mathrm{LP}} = \frac{i g_s \mathbf{T}^B}{n_{1-}\cdot n_{3-}} \left[\frac{g_W y_t}{2(n_{1+}\cdot q_1)}\right]^2 \frac{1}{r\bar{r}} \Big(\bar{v}_{c_1}(\bar{r}q_1)\frac{\not{n}_{1+}}{2} u_{c_1}(rq_1) n_{3-\nu}\varepsilon^{\nu}_{\perp_1}(q_2)$$
$$+\bar{v}_{c_1}(\bar{r}q_1)\frac{\not{n}_{1+}}{2}\gamma_5 u_{c_1}(rq_1) n^{\mu}_{3-} i\epsilon^{\perp_1}_{\mu\nu}\varepsilon^{\nu}_{\perp_1}(q_2)\Big)$$



$$+ \ldots = C^{\mathrm{NLP}}_{[i]}(s,t;\,r)$$

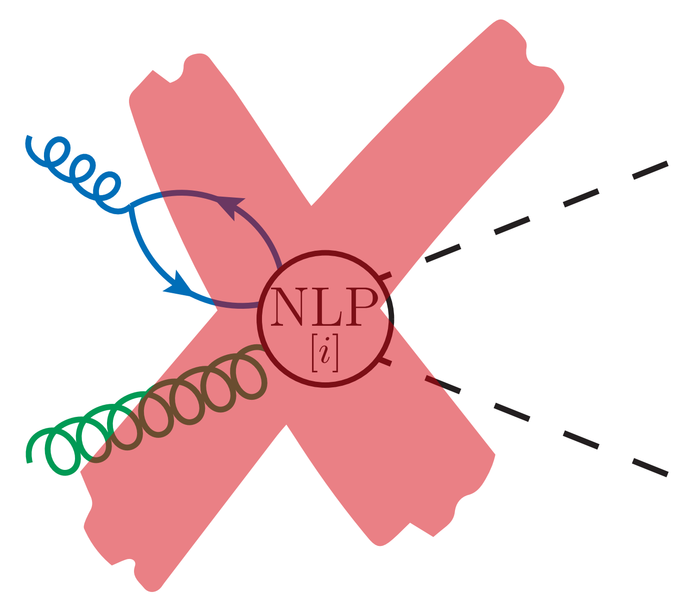


- Operator structures: “scalars”

$$J_{S_i}(\{t_{i_1},t_{i_2}\}) = \bar{\chi}_{c_i}(t_{i_2}n_{i+})\frac{\not{n}_{i+}}{2}\,\chi_{c_i}(t_{i_1}n_{i+}),$$

$$J_{P_i}(\{t_{i_1},t_{i_2}\}) = \bar{\chi}_{c_i}(t_{i_2}n_{i+})\frac{\not{n}_{i+}}{2}\,\gamma_5\,\chi_{c_i}(t_{i_1}n_{i+}),$$

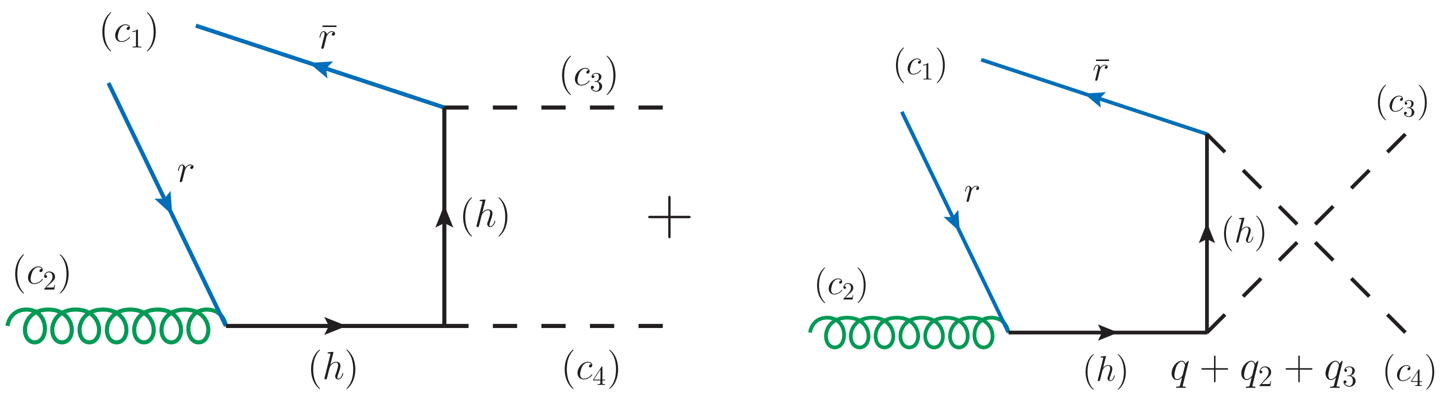
Mixing with the external gluon is forbidden



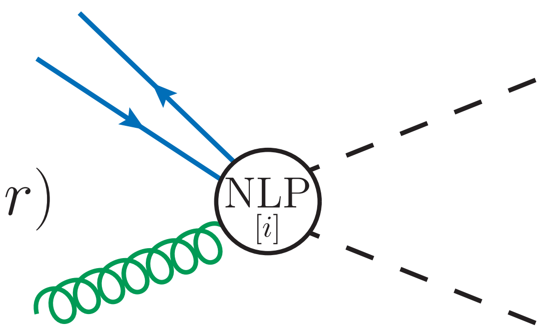
HARD SCALE $\mu^2 \sim s$

Sub-sub-leading power: multiple hard operators allowed

$$\mathcal{M}^{\rm QCD}_{\rm NLP} \sim i g_s \mathbf{T}^B \left[\frac{g_W y_t}{2(n_{1+}\cdot q_1)}\right]^2 \frac{1}{\bar{r}^2 r^2} \frac{(2\, m_t)}{n_{1+}\cdot q_1} \frac{n_{4-}\cdot n_{3-}}{n_{1-}\cdot n_{4-}} \frac{1}{n_{1-}\cdot n_{3-}}$$
$$\times \bar{v}_{c_1}(\bar{q}_1) \frac{\not{n}_{1+}}{2} \gamma_{\perp_1 \nu} u_{c_1}(rq) \varepsilon^{\nu}_{\perp_1}(q_2)$$



$$+\ldots = C^{\rm NLP}_{[i]}(s,t;\,r)$$



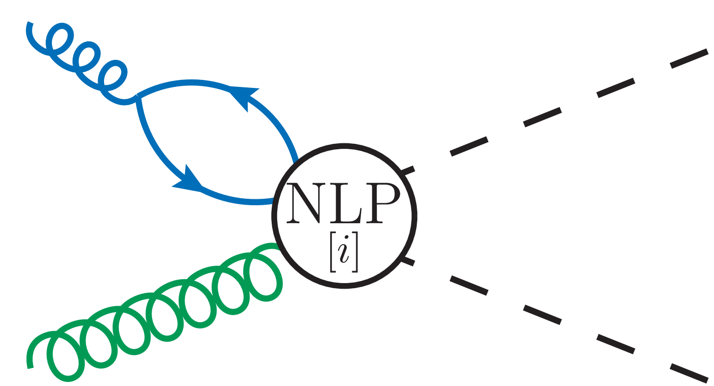
- Operator structures

$$J^{[1]}_{\rm NLP}(\{t_{1_1},t_{1_2}\},t_2,t_3,t_4) = y_t^2\; m_t\; J^{\nu\, A}_{V_1}(\{t_{1_1},t_{1_2}\})\; \mathscr{A}^{c_2\perp_2\, A}_{\nu}(t_2 n_{2+})\; h_{c_3}(t_3 n_{3+})\; h_{c_4}(t_4 n_{4+})$$

with

$$J^{\nu\, A}_{V_i}(\{t_{i_1},t_{i_2}\}) = \bar{\chi}_{c_i}(t_{i_2} n_{i+}) \frac{\not{n}_{i+}}{2} \gamma^{\nu}_{\perp_i} \mathbf{T}^A \chi_{c_i}(t_{i_1} n_{i+})$$

This contribution can mix with the external gluon

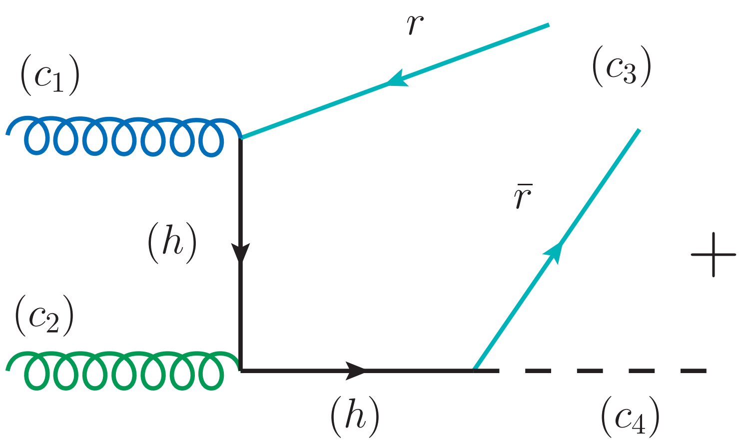


Tree-level matching $\otimes$ one loop SCET II matrix element

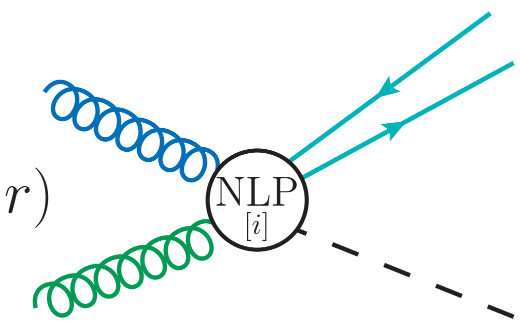
HARD SCALE $\mu^2 \sim s$

What about mixing with the Higgs?

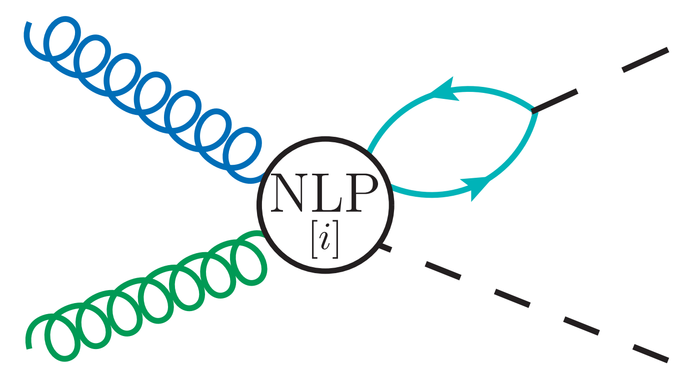
$$\mathcal{M}^{\rm QCD}_{\rm LP} \sim i g_s^2 \mathbf{T}^B \mathbf{T}^A \left[\frac{g_W y_t}{2}\right] \bar{v}_{c_3}(q') \frac{1}{\bar{r}(n_{3+} q_3)} \left[\frac{2 n_{3-\mu}}{(n_{1+}q_1) n_{3-}\cdot n_{1-}} \frac{\not{n}_{3+}}{2}\gamma_{\nu\perp_3} u_{c_3}(q) + \frac{1}{r(n_{3+}q_3)} n_{3+\nu}\frac{\not{n}_{3+}}{2}\gamma_{\mu\perp_3}u_{c_3}(q)\right]\varepsilon^{\nu}_{\perp_2}(q_2)\ \varepsilon^{\mu}_{\perp_1}(q_1)$$



$$+ \ldots = C^{\rm NLP}_{[i]}(s, t; r)$$



- Here the situation is reversed. The structures appearing at the first sub-leading power are vector type, which cannot mix with the scalar Higgs boson



- Collinear region (3): sub-sub-leading terms give non-vanishing contributions

SCALE $\mu^2 \sim m_t^2$

Now, we have to integrate out the top quark

$$\mathcal{J}^{[i]}_{\mathrm{LP}}(t_1, t_2) = P^{\mu\nu}_{[i]} \mathcal{A}^{\mathrm{PDF-c_1\perp_1}}_{\mu}(t_1 n_{1+}) \mathcal{A}^{\mathrm{PDF-c_2\perp_2}}_{\nu}(t_2 n_{2+})$$

We take massive SCET (nf=6) matrix elements and match all the contributions at massless SCET (nf=5)

$$\left\langle H(q_3)H(q_4)\right| J^{[i]}_{\mathrm{LP}}(0,0,0,0)\left| g(q_1)g(q_2)\right\rangle = \mathcal{C}\,\langle 0 |\, \mathcal{J}^{[i]}_{\mathrm{LP}}(0,0)\left| g(q_1)g(q_2)\right\rangle$$

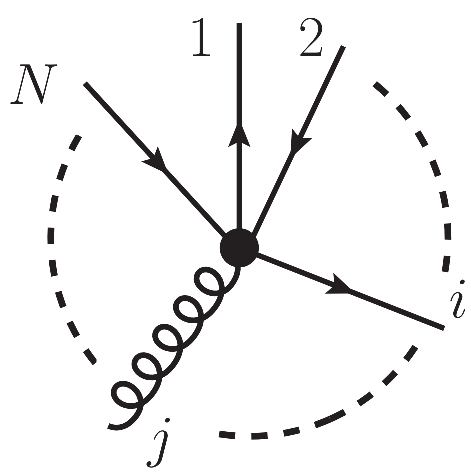
This matching is not trivial, but the leading power m_t^2 dependence is universal and can be restored via "massification"

[A. Mitov, A. Moch, hep/0612149] [T. Becher, K. Melnikov, 0704.3582] [G. Wang, T. Xia, L. L. Yang, X. Ye, 2312.12242]

$$\left|\mathcal{M}^{\mathrm{massive}}(\{p\},\{m\})\right\rangle = \prod_i \left(\mathcal{Z}^{(m|0)}_{[i]}(\{m\})\right)^{1/2} \boldsymbol{\mathcal{S}}(\{p\},\{m\}) \left|\mathcal{M}^{\mathrm{massless}}(\{p\})\right\rangle$$

MASS EFFECTS (MASSIFICATION)

EFT operator matched on the on-shell amplitude at arbitrary loop order



Hard region only $\mathscr{A}(m=0) = C(\{p_i\})\left\langle \prod_k \psi_k(0) \right\rangle$

EFT renormalization

$$C(\{p_i\}, \epsilon) = Z(\{p_i\}, \epsilon, \mu) C(\{p_i\}, \mu)$$

In QCD language: **IR poles** **finite remainder**

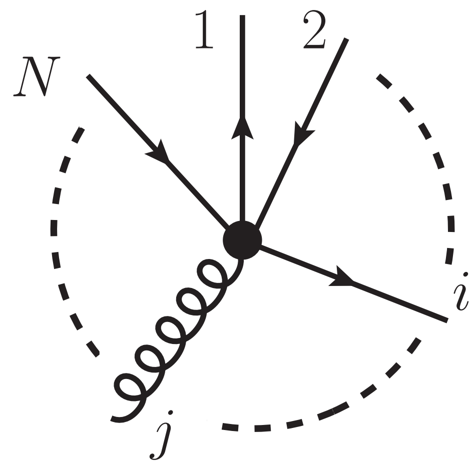
In massless theory,
the bare matrix element:

$$\left\langle \prod_k \psi_k(0) \right\rangle = \prod_k u_k(p_k)$$

In massive theory,
the bare matrix element:

$$\left\langle \prod_k \psi_k(0) \right\rangle = S(m, \epsilon) \prod_k j_k(m, \epsilon) u_k(p_k)$$

MASS EFFECTS (MASSIFICATION)



In massive theory, the bare matrix element:

$$\left\langle \prod_k \psi_k(0) \right\rangle = S(m,\epsilon)\prod_k j_k(m,\epsilon)u_k(p_k)$$

Renormalization:

$$j_k(m,\epsilon) = Z_k(m,\mu,\epsilon)j_k(m,\mu) \qquad S(m,\epsilon) = Z_s(m,\mu,\epsilon)S(m,\mu)$$

Finite remainder must be the same, so it allows us to derive relation

$$\mathscr{A}(m \neq 0) = Z^{-1}(\{p_i\},\epsilon,\mu)Z_s(m,\mu,\epsilon)\prod_k Z_k(m,\mu,\epsilon)\mathscr{A}(m=0)$$

In QCD: a method to restore leading mass dependence

In SCET:

- can be generalized to all orders and resummed
- can be generalized to leading power suppressed terms

SCALE $\mu^2 \sim m_t^2$

- At the scale m_t^2, the heavy top quarks must be integrated out

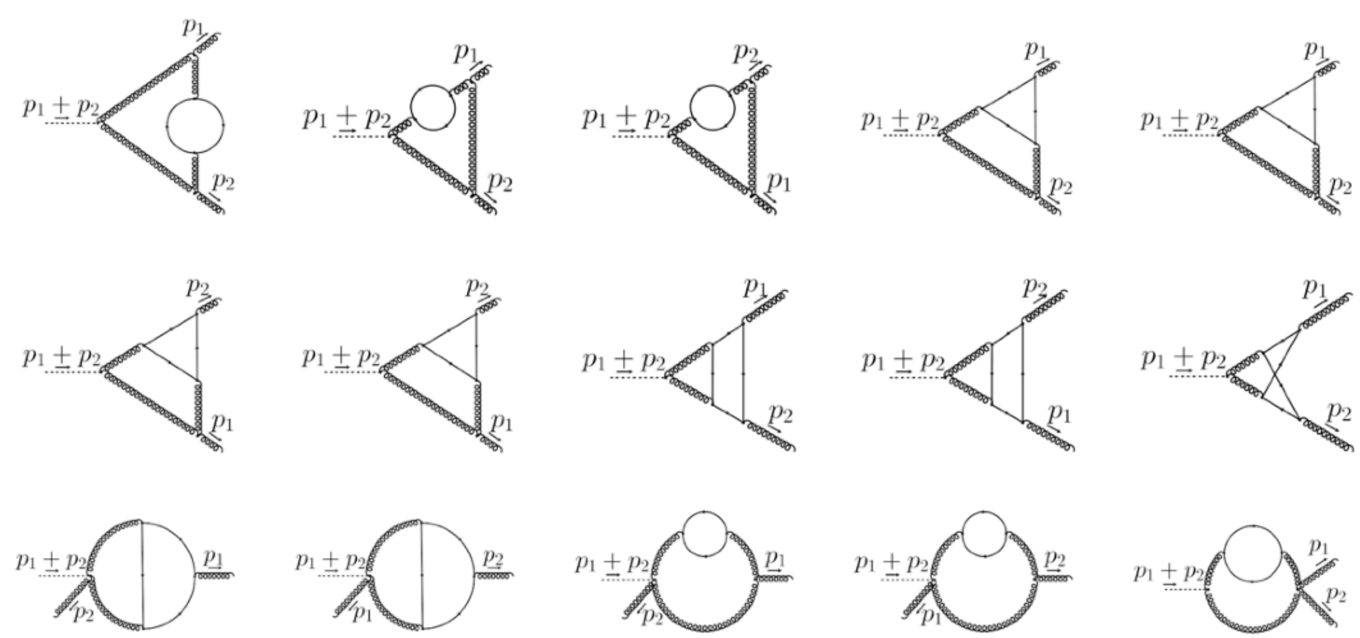
$$\left|\mathcal{M}^{\text{massive}}(\{p\},\{m\})\right\rangle = \prod_i \left(\mathcal{Z}_{[i]}^{(m|0)}(\{m\})\right)^{1/2} \boldsymbol{\mathcal{S}}(\{p\},\{m\}) \left|\mathcal{M}^{\text{massless}}(\{p\})\right\rangle$$

For the purpose of $gg \to HH$ we just use "massification" to obtain th

matching coefficient

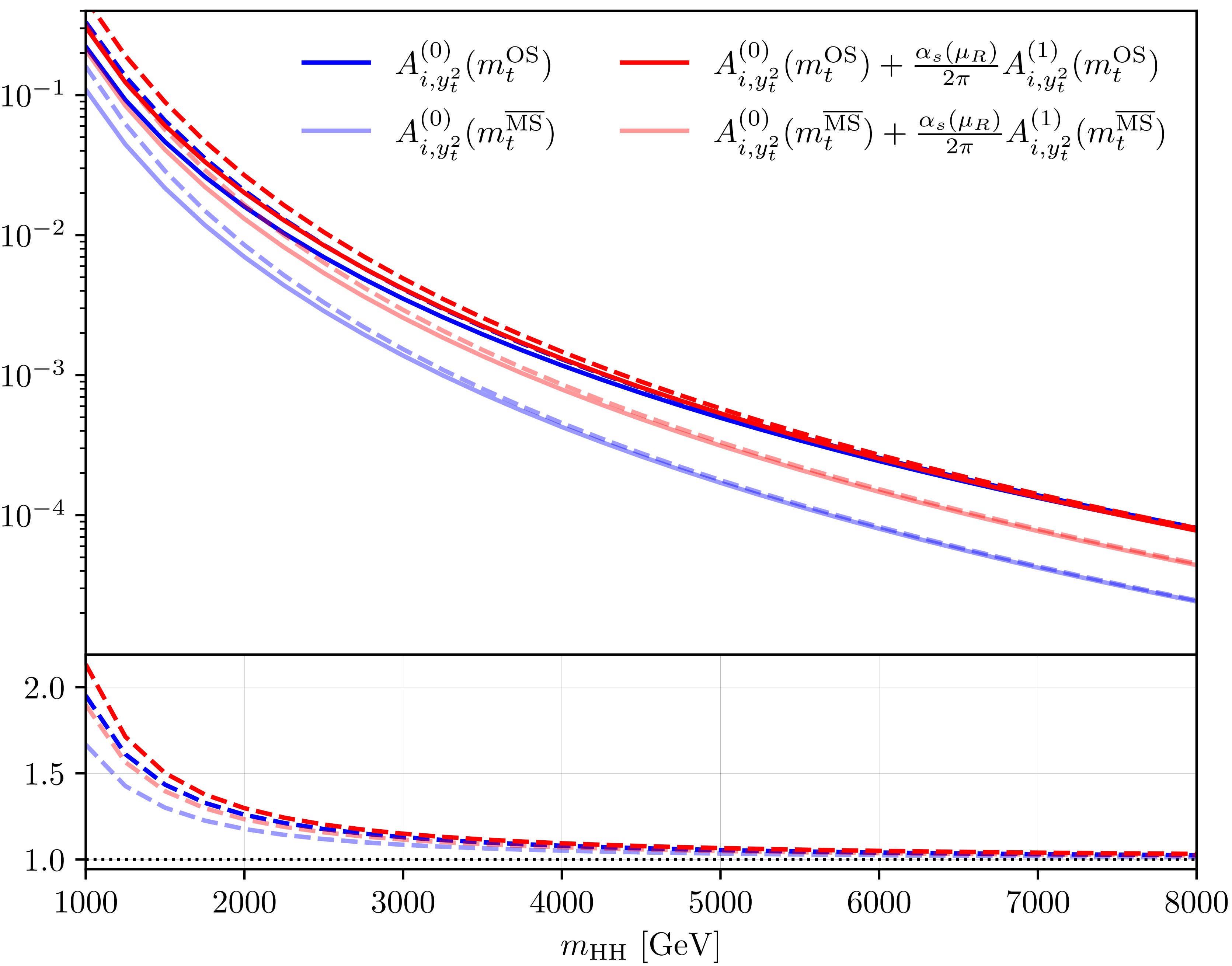
$$\mathcal{S}(\{p\}, m_t) = 1 + C_A T_f \left(\frac{\alpha_s^{(n_f=6)}}{4\pi}\right)^2 \left(\frac{\mu^2}{m_t^2}\right)^{2\epsilon} \left(-\frac{4}{3\epsilon^2} + \frac{20}{9\epsilon} - \frac{112}{27} - \frac{4\zeta_2}{3}\right) \ln\frac{-s}{m_t^2}$$

$$\mathscr{C} = Z_{(n_f=5)}^{-1} Z_{(n_f=6)} \mathscr{Z}_g^{(m|0)}(m_t)\, \mathcal{S}(\{p\}, m_t)$$



$$\mathscr{C} = 1 + \left(\frac{\alpha_s}{4\pi}\right)^2 \left(\frac{112}{27} C_A T_F \ln\left(-\frac{m_t^2}{s}\right) - \frac{28}{9} T_F C_A \zeta_3 - \frac{5}{27}\pi^2 T_F C_A + \frac{262}{27} T_F C_A\right)$$

RESULTS: HIGH ENERGY EXPANSION



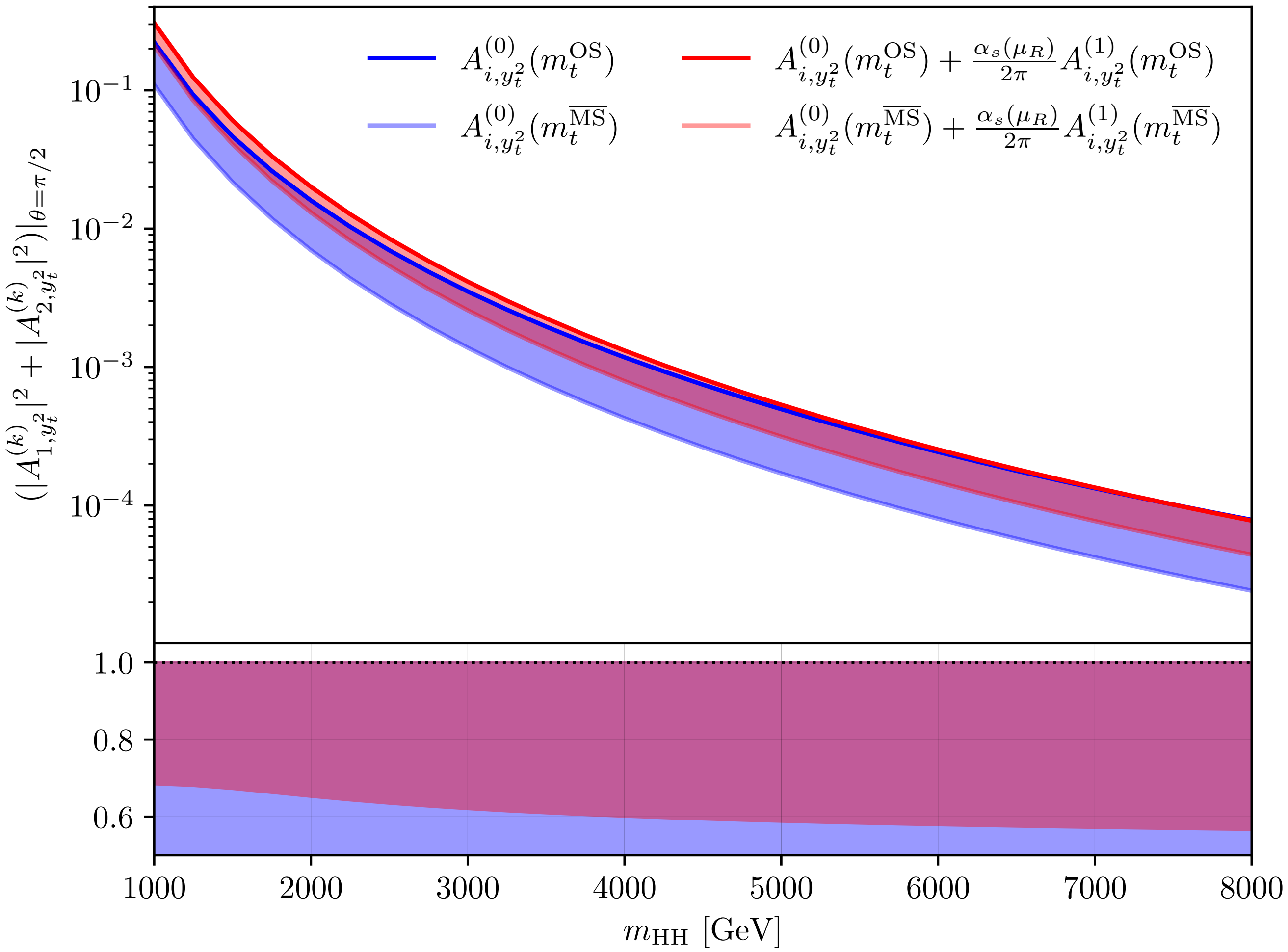
EXPANSION VALIDITY

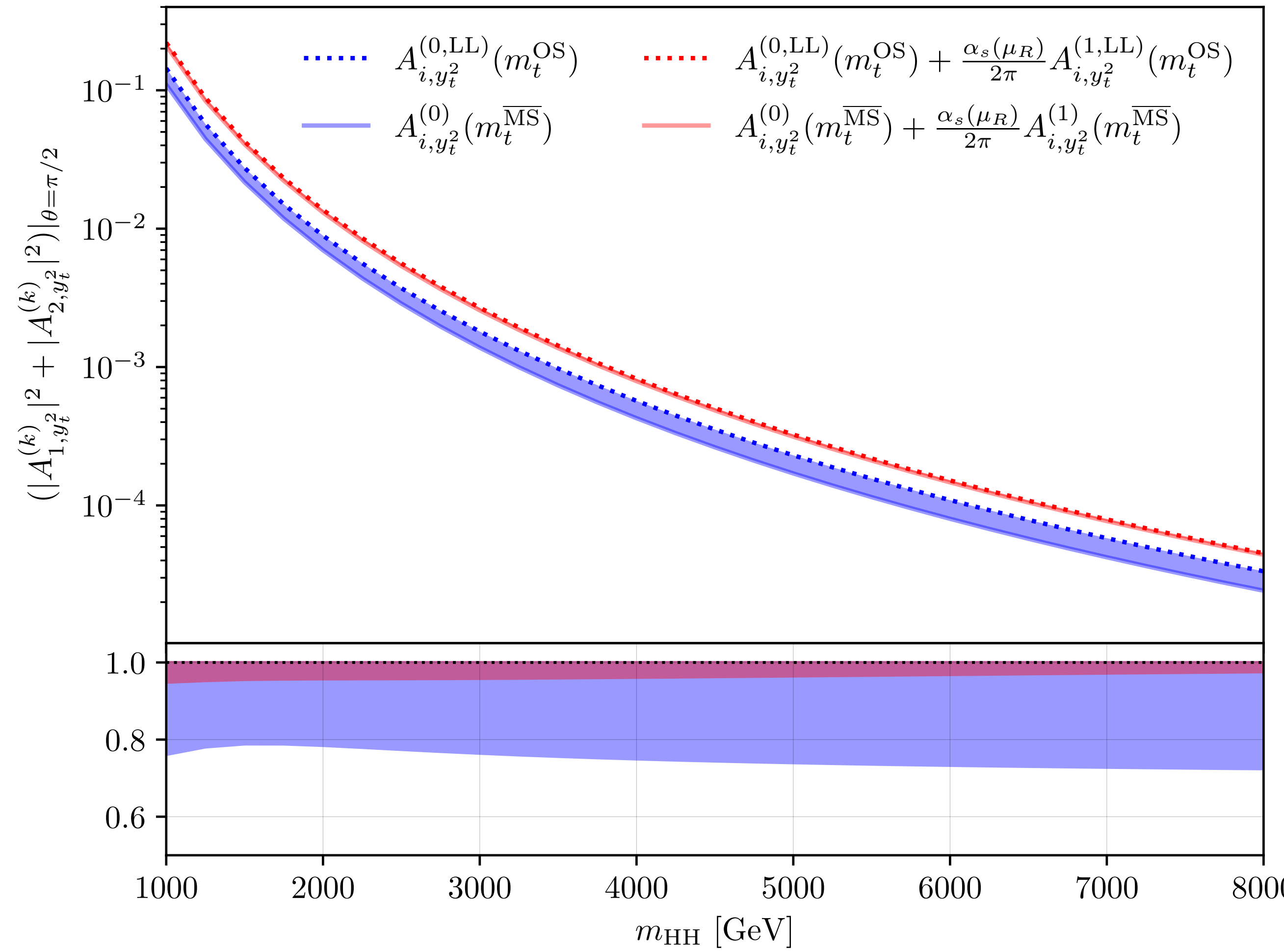
Solid lines: full results

Dashed lines: LP

Power expansion converges in the high-energy limit!

RESULTS: UNCERTAINTY





Traditional approach: uncertainty dominated by the scheme conversion at fixed order

EFT improved approach: LP logarithms are resumed to all orders, and uncertainty comes from NLP (this could be further improved)

NEXT-TO-LEADING POWER EXAMPLE:

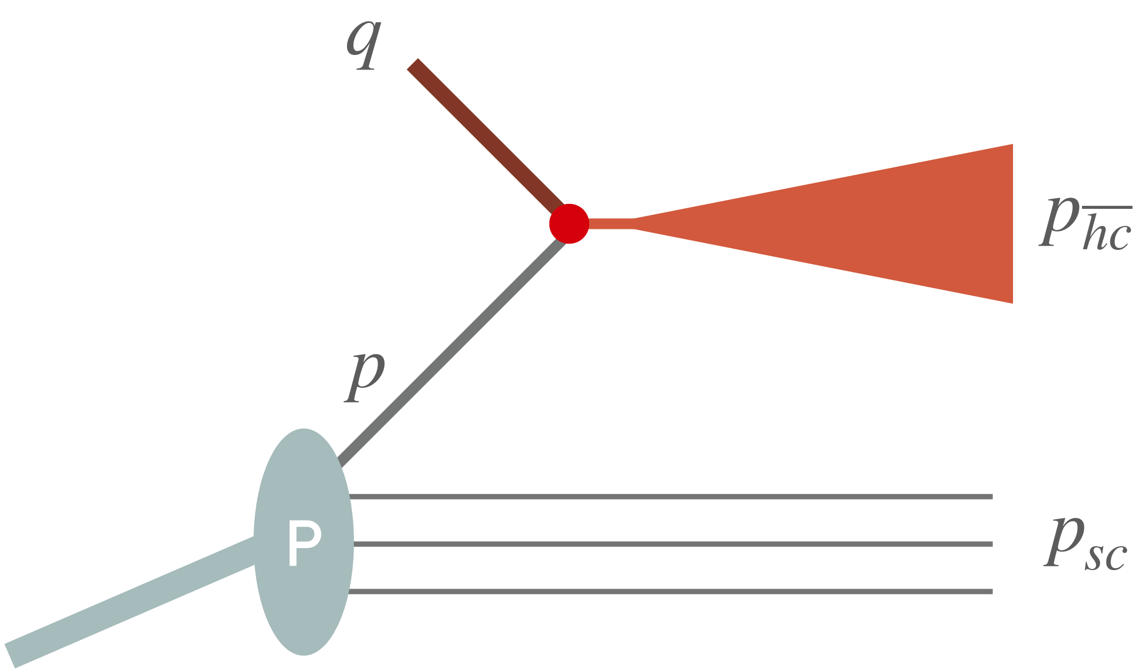
THRESHOLD PARTON DISTRIBUTION FUNCTIONS BEYOND LEADING POWER

[25…, with Martin Beneke]

[25…, with Martin Beneke and Marvin Schnubel]

DIS NEAR THRESHOLD

We focus on DIS in the $x \to 1$ limit



Proton with momentum P is struck by a probe with momentum q

$$p + q = P_X \qquad x = \frac{Q^2}{2pq} \qquad Q^2 = -q^2$$

The final state has small virtuality for $x \to 1$

$$P_X^2 = 2pq - Q^2 = Q^2 \frac{1-x}{x}$$

We have to consider final state jet and proton remnants

Final state: $P_X = p_{\overline{hc}} + p_{sc}$

Anti-hard-collinear jet + *Proton remnants*

KINEMATICS

$$W^{\mu\nu}(p,q) = \frac{1}{4\pi}\sum_X \int dPS_X\,(2\pi)^d\,\delta^{(d)}\left(p+q-p_X\right)\left\langle N(p)\,|\,J^{\dagger\nu}(0)\,|\,X\right\rangle\langle X\,|\,J^{\mu}(0)\,|\,N(p)\rangle = -\,g^{\mu\nu}F_1(Q^2,x)+\ldots$$

We choose light-cone momenta to parameterize the initial state

$$p^\mu = n_+p\frac{n_-^\mu}{2}, \qquad q^\mu = -\,Q\frac{n_-^\mu}{2} + Q\frac{n_+^\mu}{2}\,,$$

Scaling parameters

$$\lambda = \sqrt{1-x}, \qquad \xi = \frac{\Lambda_{\mathrm{QCD}}}{Q}$$

We work to LP in ξ and NLP in λ

Modes $(n_+p, p_\perp, n\_p)$

Collinear (struck quark): $Q(1,\xi,\xi^2)$

Anti-hardcollinear (jet): $Q(\lambda^2,\lambda,1)$

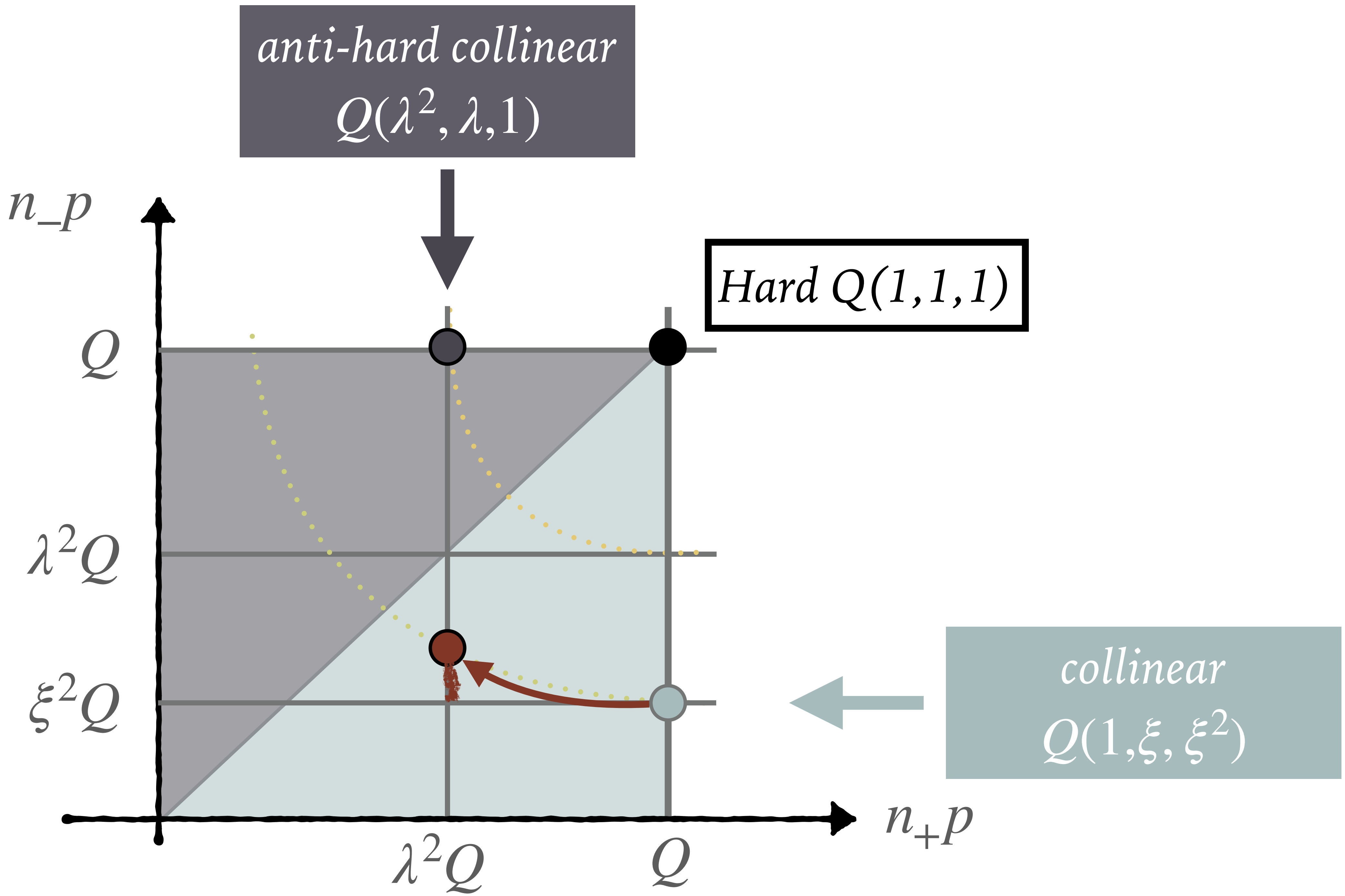
Soft-collinear (remnant): $Q(\lambda^2,\xi,\frac{\xi^2}{\lambda^2})$

Leading twist & perturbative power corrections

Compare:

[A. Hoang et. al., 1508.04323]

THE PROTON REMNANT



Kinematics restricts n_+p components of the remnants to $\lambda^2 Q$

Soft-collinear (remnant): $Q(\lambda^2, \xi, \frac{\xi^2}{\lambda^2})$

THRESHOLD PDF

At Leading power, the PDF factorizes

$$f_{q/N}(\xi) \to \int \frac{dt}{2\pi}\, e^{i(1-\xi)tn_+p}\, |C_\Lambda|^2 \langle N(p)|\bar{\chi}^{(0)}_{fc}[Y^\dagger_{sc,n_-}\overline{Y}_{sc,n_+}](tn_+)\frac{\not{n}_+}{2} \times [\overline{Y}^\dagger_{sc,n_+}Y_{sc,n_-}](0)\chi^{(0)}_{fc}|N(p)\rangle$$

$$= n_+p\, K_{q/N}\, |C_\Lambda|^2\, S^{\rm LP}_{sc}(\frac{Q(1-\xi)}{x})$$

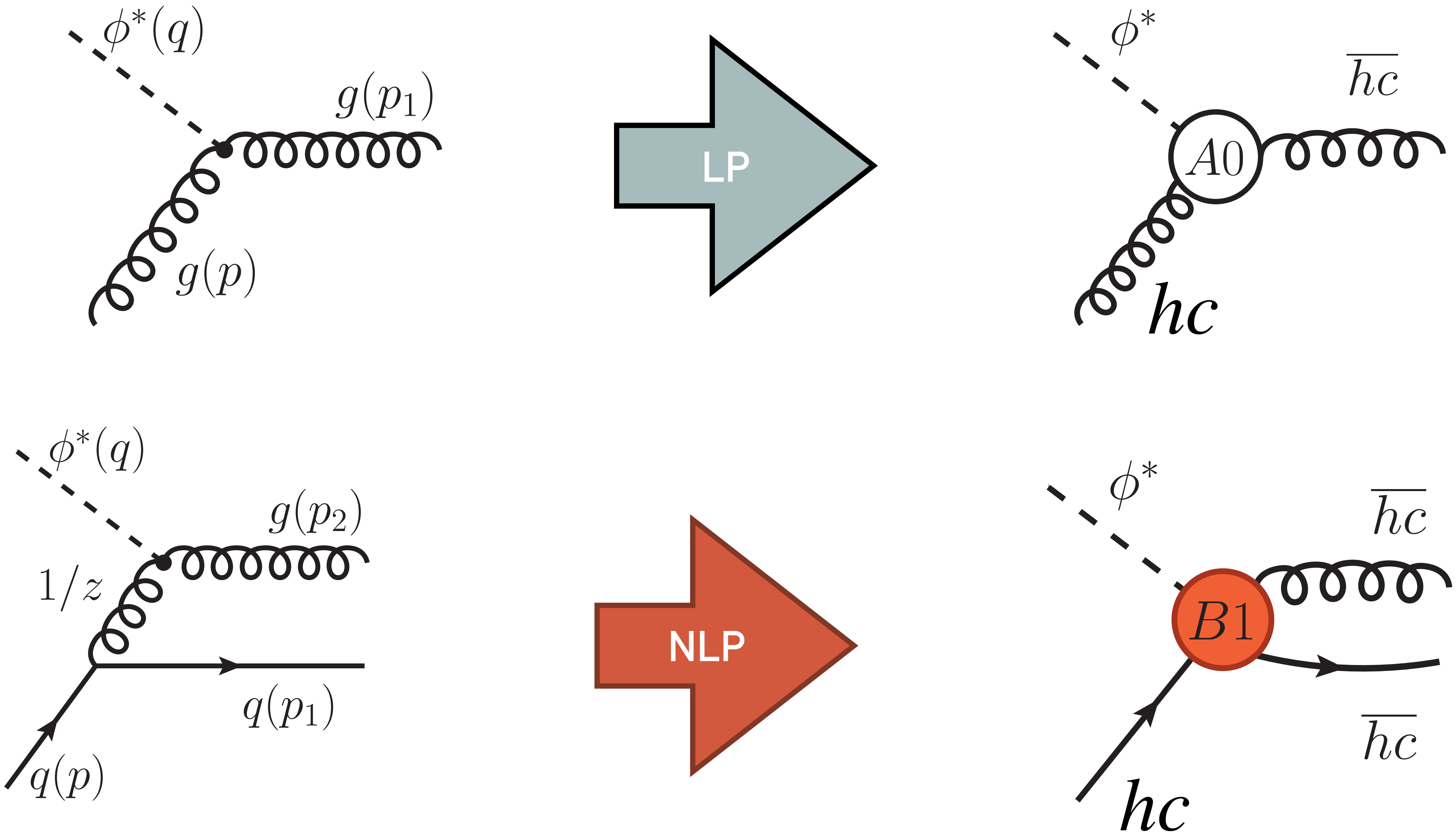
$|C_\Lambda|^2$: matching of pdf-collinear fields on frozen-collinear

Non-perturbative analog of $C_m(m_e;\mu) = 1 + \frac{\alpha}{4\pi}\left\{2\ln^2\left(\frac{m_e}{\mu}\right) - \ln\left(\frac{m_e}{\mu}\right) + \frac{\pi^2}{12} + 2\right\}$

$S^{\rm LP}_{sc}(\frac{Q(1-\xi)}{x})$: soft-collinear function

NEXT-TO-LEADING POWER COLLINEAR CONTRIBUTION

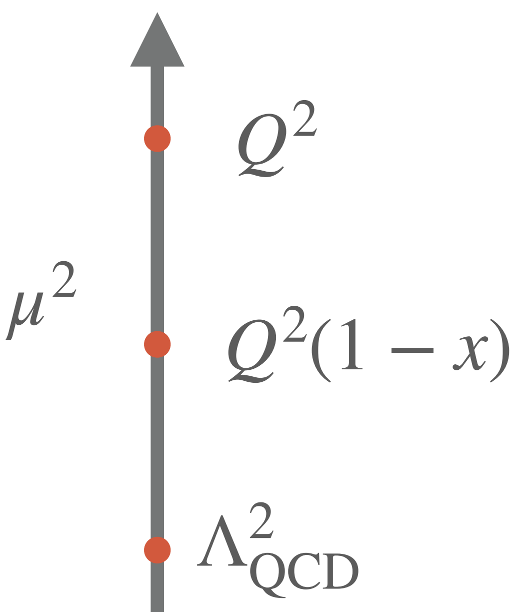
We will look at off-diagonal channel



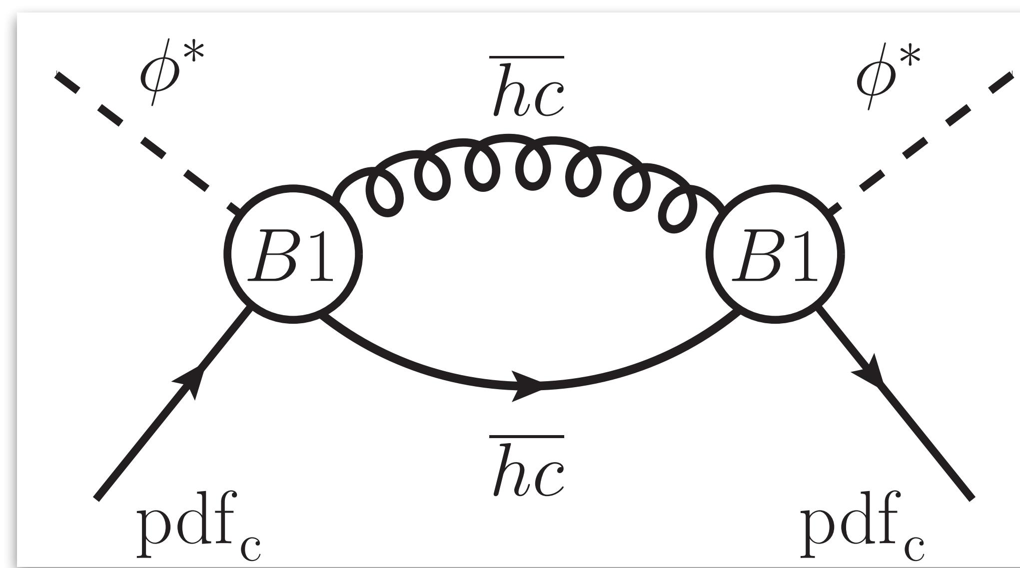
NLP FACTORIZATION: HARD-COLLINEAR PART

Recall: $$W^{\mu\nu} = -\, g_\perp^{\mu\nu} \left| C^{A0}(n_+p, Q) \right|^2 Q^2 \int_x^1 \frac{d\xi}{2\pi} \frac{1}{x} \mathcal{J}_{\overline{hc}}^{(q)} \left(Q^2 \frac{\xi - x}{x} \right) n_+p\, K_N S_{sc}^{\mathrm{LP}} \left(\frac{Q(1-\xi)}{x} \right)$$

$$W_\phi = \frac{1}{2Q^2} \int_0^1 drdr'\, C^{\mathrm{B1}*}(r') C^{\mathrm{B1}}(r) \times \int_x^1 \frac{d\xi}{2\pi} \frac{1}{x} \mathcal{J}_{\overline{hc}}^{(qg)} \left(Q^2 \frac{\xi - x}{x}, r, r' \right) n_+p\, K_N S_{sc}^{\mathrm{LP}} \left(\frac{Q(1-\xi)}{x} \right)$$



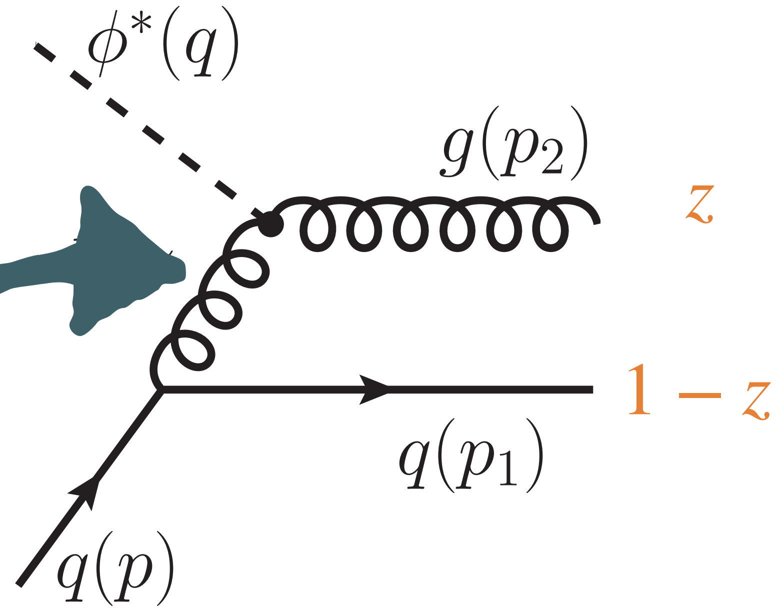
Anomalous dimension of $\mathcal{J}_{\overline{hc}}^{(qg)}$ can be fixed from consistency relations, but:

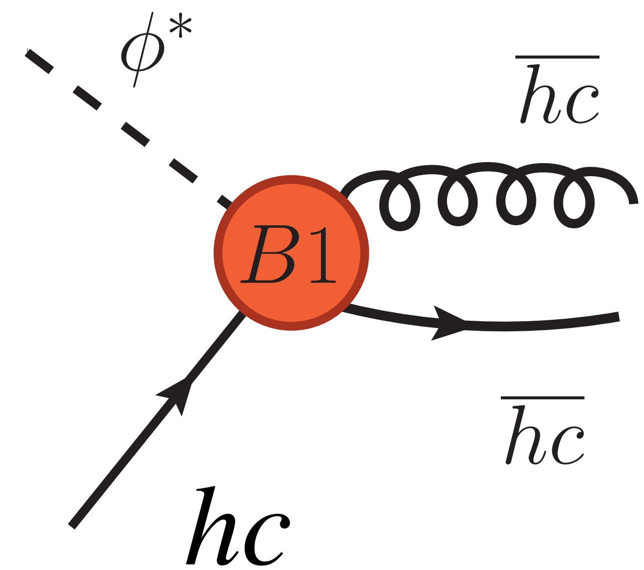


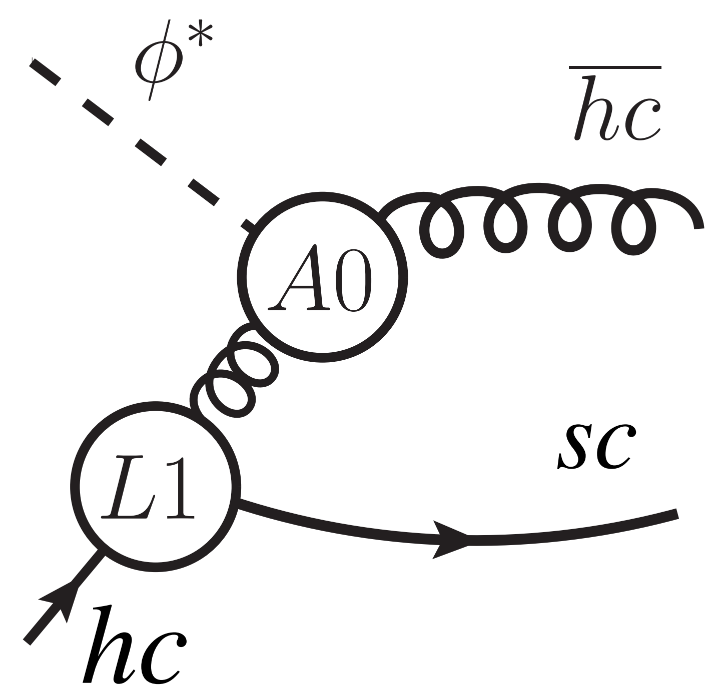
$$\boxed{\int_0^1 drdr'\, C^{\mathrm{B1}*}(r') C^{\mathrm{B1}}(r) \mathcal{J}_{\overline{hc}}^{(qg)} \left(Q^2 \frac{\xi - x}{x}, r, r' \right) \sim \int_0^1 dr \frac{1}{1-r}}$$

PHYSICAL ORIGIN OF THE ENDPOINT DIVERGENCE

For $z \sim 1$ intermediate propagator is hard







$$\boxed{(p-p_1)^2 = -2p\cdot p_1 \approx -n_+p n_-p_2 \sim (1-z)Q^2 \ll Q^2}$$

For $\bar{z} \equiv 1-z \ll 1$ soft and collinear modes overlap

THE SOFT-COLLINEAR CONTRIBUTION

$$\Big\langle X \Big| \big[G^{A}_{\mu\nu} G^{\mu\nu A}\big](0) \Big| N(p) \Big\rangle = \int dt d\bar{t}\, \tilde{C}^{A0}_{g}(t,\bar{t})\, 2 g^{\perp}_{\mu\nu}\, \langle X_{\overline{hc}} \,|\, n_{-}\partial \mathcal{A}^{(0)\nu B}_{\overline{hc}\perp}(\bar{t} n_{-}) \,|\, 0 \rangle$$

$$\times\, i \int d^4 y\, \langle X_{sc} \,|\, T\Big(n_{-}\partial \mathcal{A}^{\mu A}_{hc\perp}(t n_{+}) \mathcal{Y}^{AB}_{n_{+},sc}(0), \mathcal{L}^{(1)}(y) \Big) \,|\, N(p) \rangle$$

After squaring the matrix element, the first line becomes the LP gluon jet function

$$\frac{1}{2\pi} \frac{1}{g_s^2} \sum_{X_{\overline{hc}}} \int PS_{X_{\overline{hc}}} \langle 0 \,|\, \mathcal{A}^{(0)\nu B}_{\overline{hc}\perp}(x) \,|\, X_{\overline{hc}} \rangle \langle X_{\overline{hc}} \,|\, \mathcal{A}^{(0)\nu' C}_{\overline{hc}\perp}(0) \,|\, 0 \rangle \equiv \delta^{BC} (-g^{\nu\nu'}_{\perp}) \int \frac{d^d k}{(2\pi)^d}\, e^{-ikx} \mathcal{J}^{(g)}_{\overline{hc}}(k^2)$$

Following the usual procedure, we arrive at

$$W_{\phi} = \Big| C^{A0}_{g}(n_{+}p, Q) \Big|^2 \int_x^1 \frac{d\xi}{2\pi} \frac{1}{x}\, \mathcal{J}^{(g)}_{\overline{hc}}\left(Q^2 \frac{\xi - x}{x} \right)\, \mathcal{S}^{\rm NLP}_{sc}\left(\frac{Q(1-\xi)}{x} \right)$$

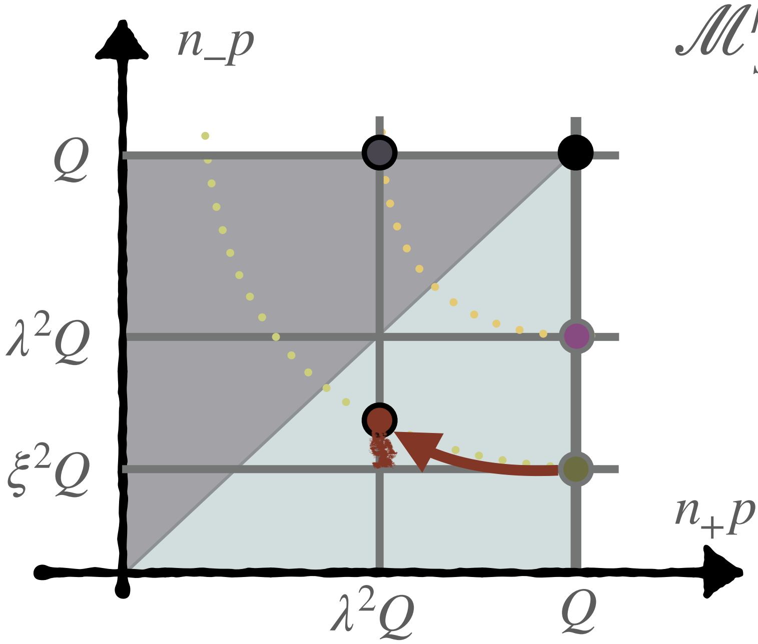
And we recognize the NLP threshold PDF contribution

$$\boxed{ f_{g/N}(x) \Big|_{x\to 1} = f^{\rm thr,LP}_{g/N}(x) + f^{\rm thr,NLP}_{g/N}(x) }$$

But is it the threshold limit of the standard PDF?

THE SOFT-COLLINEAR MATRIX ELEMENT

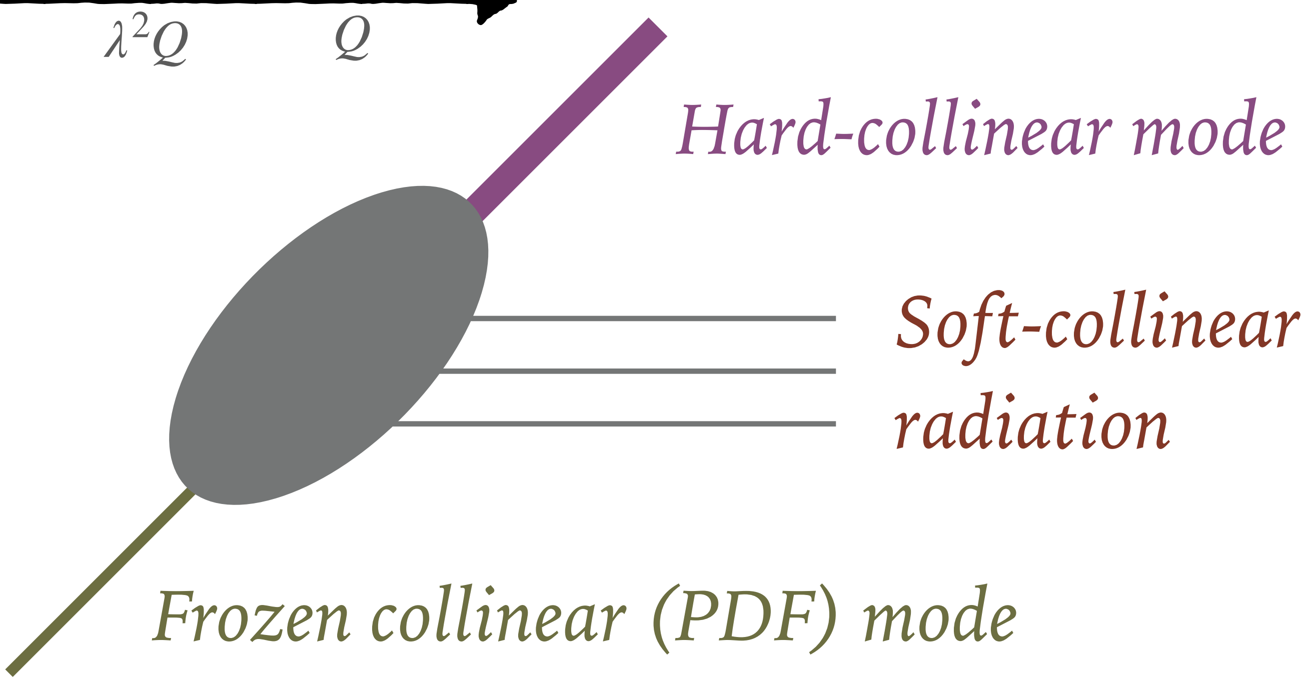
$$\mathcal{S}_{sc}^{\rm NLP}(\Omega) = \int du\, e^{iu\Omega} \sum_{X_{sc}} \int PS_{X_{sc}}\, e^{-in_+\cdot p_{X_{sc}} u}\, (-1)\, g_s[\mathcal{M}^B_{sc,\mu_\perp}(0)]^*\, g_s \mathcal{M}^{\mu_\perp B}_{sc}(0)$$

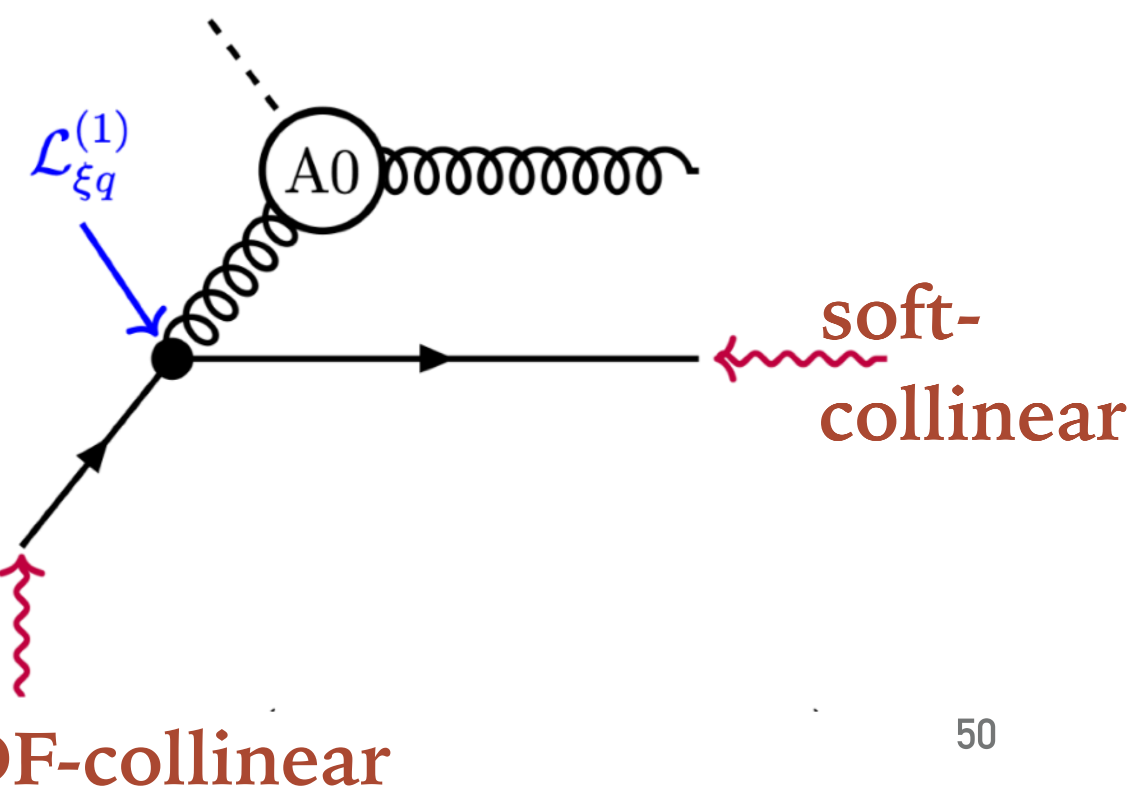


$$\mathcal{M}^{\mu_\perp B}_{sc}(t) \equiv i \int d^4y\, \langle X_{sc}\,|\, T\left(n_-\partial \mathcal{A}^{\mu A}_{hc\perp}(tn_+)\mathcal{Y}^{AB}_{n_+,sc}(0), \mathcal{L}^{(1)}(y)\right)|\,N(p)\rangle$$

Lagrangian:

$$\mathcal{L}^{(1)}(y) = [\bar\eta_{sc} Y_{sc,n_-}](y_-)\, \cancel{\mathcal{A}}^{(0)}_{hc\perp}(y)\chi_c^{(0)}(y) + \mathrm{h.c.}\,..$$





RADIATIVE JET FUNCTION

$$\boxed{J(n_+p_{hc}, n_+p_c;\omega) = \delta(n_+p_{hc} - n_+p_c)\,\frac{n_+p_c}{p^2}\,D_g^{\mathrm{B1}}(p^2)\,, \qquad p^2 = n_+p_c\omega}$$

$$\frac{d}{d\ln\mu}D_g^{\mathrm{B1}}(p^2) = \int_0^\infty d\hat{p}^2\,\gamma_{D_g}(\hat{p}^2, p^2)D_g^{\mathrm{B1}}(\hat{p}^2)\,,$$

New renormalziation kernel

$$\gamma_{D_g}(\hat{p}^2, p^2) = \frac{\alpha_s(C_A - C_F)}{\pi}\,\delta(\hat{p}^2 - p^2)\ln\left(\frac{\mu^2}{-p^2 - i\varepsilon}\right) - \frac{\alpha_s}{\pi}\frac{C_A}{2}p^2\left[\frac{\theta(\hat{p}^2 - p^2)}{\hat{p}^2(\hat{p}^2 - p^2)} + \frac{\theta(p^2 - \hat{p}^2)}{p^2(p^2 - \hat{p}^2)}\right]_+$$
$$-\frac{3}{2}\frac{\alpha_s}{\pi}C_F\delta(\hat{p}^2 - p^2)$$

The anomalous dimensions of radiative jet functions are universal up to color factors

THE SOFT-COLLINEAR FUNCTION (NLP ENDPOINT PDF)

Non-perturbative matrix elements

$$\mathcal{S}^{\mathrm{NLP}}_{sc}(\Omega) = (d-2)\, n_+p\, K_N \int_0^\infty d\omega d\omega'\, D_g^{\mathrm{B1}}(\omega) D_g^{B1}(\omega')\, S^{\mathrm{NLP}}_{sc}(\Omega,\omega,\omega')$$

The soft-collinear matrix element becomes

$$S^{\mathrm{NLP}}_{sc}(\Omega,\omega,\omega') = g_s^2 \int du\, e^{iu\Omega} \int dv\, e^{-i\omega v} \int dv'\, e^{i\omega' v'} \times \langle 0 |\, tr \left(\mathcal{O}^{\dagger B}_{sc}([u+v]n_-, un_-) \frac{\not{n}_-}{2}\, \mathcal{O}^{B}_{sc}(vn_-,0) \right) | 0 \rangle$$

With the soft function containing explicit soft quark field

$$\mathcal{O}^{B}_{sc,\alpha a}(vn_-,0) == T\left(\left[\bar{\eta}_\alpha(vn_+) T^C Y_{sc,n_-}(vn_-) \right]_a [vn_-,0]^{CA}_{sc,n_-}\, \mathcal{Y}^{AB}_{sc,n_+}(0) \right)$$

Soft quark field *Finite distance Wilson line* *Adjoint Wilson line*

TWO WAYS TO DEAL WITH THE ENDPOINT DIVERGENCE

Stay in d-dimensions

[M. Beneke, M. Garny, S. Jaskiewicz, **RS**, L. Vernazza, J. Wang: 2008.04943]

$$\left[C^{A0}\left(zQ^2,\mu^2\right)\right]_{\text{bare}} = C^{A0}\left(Q^2,Q^2\right)\exp\left[-\frac{\alpha_s C_A}{2\pi}\frac{1}{\epsilon^2}\left(\frac{Q^2}{\mu^2}\right)^{-\epsilon}\right]$$

$$\left[D^{B1}\left(zQ^2,\mu^2\right)\right]_{\text{bare}} = D^{B1}\left(zQ^2,zQ^2\right)\exp\left[-\frac{\alpha_s}{2\pi}\left(C_F-C_A\right)\frac{1}{\epsilon^2}\left(\frac{zQ^2}{\mu^2}\right)^{-\epsilon}\right]$$

$$\tilde{C}^{NLP,LL}_{\phi,q}\Big|_{\epsilon\to 0} = \frac{1}{2N\ln N}\frac{C_F}{C_F-C_A}\left(\mathscr{B}_0(a)\exp\left[C_A\frac{\alpha_s}{\pi}\left(\frac{1}{2}\ln^2 N+\ln N\ln\frac{\mu^2}{Q^2}\right)\right]-\exp\left[\frac{\alpha_s C_F}{\pi}\left(\frac{1}{2}\ln^2 N+\ln N\ln\frac{\mu^2}{Q^2}\right)\right]\right)$$

$$\gamma^{NLP,LL}_{gq}(N) = -\frac{1}{N}\frac{\alpha_s C_F}{\pi}\mathscr{B}_0(a)\,,$$

$$a = \frac{\alpha_s}{\pi}(C_F-C_A)\ln^2 N$$

$$\mathscr{B}_0(x) = \sum_{n=0}^{\infty}\frac{B_n}{(n!)^2}x^n$$

In agreement with a conjecture

[A. Vogt, 1005.1606]

TWO WAYS TO DEAL WITH THE ENDPOINT DIVERGENCE

Introduce reshuffling of the endpoint

$$W_\phi \sim \int_x^1 \frac{d\xi}{2\pi}\frac{n_+p\,K_N}{x}\Bigg[\int_0^1 drdr'\,C^{\mathrm{B1}*}(r')C^{\mathrm{B1}}(r)\mathcal{J}^{(qg)}_{\overline{hc}}\left(Q^2\frac{\xi-x}{x},r,r'\right)\,S^{\mathrm{LP}}_{sc}\left(\frac{Q(1-\xi)}{x}\right)$$

$$+\left|C^{\mathrm{A0}}_g\right|^2\,\mathcal{J}^{(g)}_{\overline{hc}}\left(Q^2\frac{\xi-x}{x}\right)\int_0^\infty d\omega d\omega'\,D^{\mathrm{B1}}_g(\omega)D^{B1}_g(\omega')\,S^{\mathrm{NLP}}_{sc}\left(\frac{Q(1-\xi)}{x},\omega,\omega'\right)\Bigg]$$

Endpoint consistency conditions:

$$C^{\mathrm{B1}}(r;Q^2)\Big|_{r\to 1} = C^{\mathrm{A0}}_g(Q^2)\times\frac{D^{B1}(Q^2r)}{r}$$

$$\mathcal{J}^{(qg)}_{\overline{hc}}\left(p^2,r,r'\right)\Big|_{r,r'\to 1}S^{\mathrm{LP}}_{sc}(\Omega) = \mathcal{J}^{(g)}_{\overline{hc}}\left(p^2\right)S^{\mathrm{NLP}}_{sc}\left(\frac{Q(1-\xi)}{x},\omega,\omega'\right)\Big|_{\omega,\omega'\to\infty}$$

Compare

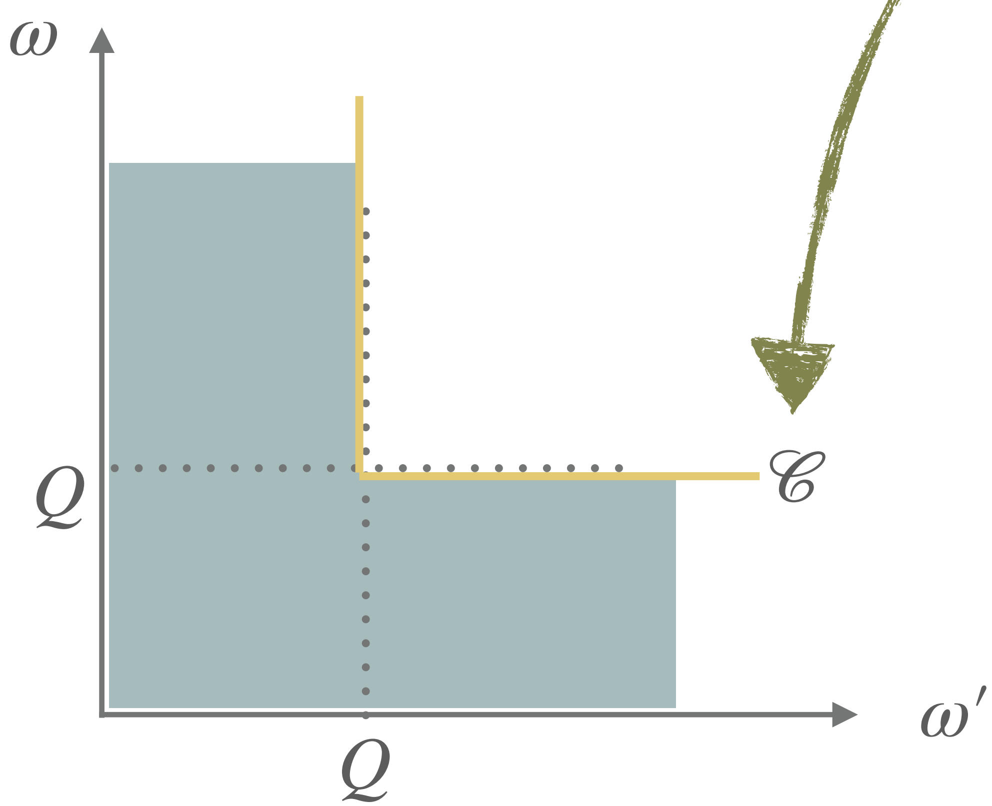
[M. Beneke, M. Garny, S. Jaskiewicz, J. Strohm, **RS**, L. Vernazza, J. Wang, 2205.04479]

Amplitude level: [M. Neubert et. al., 1912.08818, 2009.06779, 2212.10447]

$B \to X_s\gamma$: [T. Hurth, **RS**, 2301.01739]

TWO WAYS TO DEAL WITH THE ENDPOINT DIVERGENCE

$$
\begin{aligned}
W_\phi \sim \int_x^1 \frac{d\xi}{2\pi}\,\frac{n_+p\,K_N}{x}\Bigg[\int_0^1 dr dr' \Bigg\{ & C^{\mathrm{B1}*}(r')C^{\mathrm{B1}}(r)\mathcal{J}^{(qg)}_{\overline{hc}}\left(Q^2\frac{\xi-x}{x},r,r'\right)\,S^{\mathrm{LP}}_{sc}\left(\frac{Q(1-\xi)}{x}\right) \\
& C^{\mathrm{B1}*}(r')C^{\mathrm{B1}}(r)\mathcal{J}^{(qg)}_{\overline{hc}}\left(Q^2\frac{\xi-x}{x},r,r'\right)\,S^{\mathrm{LP}}_{sc}\left(\frac{Q(1-\xi)}{x}\right)\Bigg|_{r,r'\to 1}\Bigg\} \\
& + \left|C_g^{\mathrm{A0}}\right|^2 \mathcal{J}^{(g)}_{\overline{hc}}\left(Q^2\frac{\xi-x}{x}\right)\int_{\mathscr{C}} d\omega d\omega'\, D_g^{\mathrm{B1}}(\omega)D_g^{B1}(\omega')\,S^{\mathrm{NLP}}_{sc}\left(\frac{Q(1-\xi)}{x},\omega,\omega'\right)\Bigg]
\end{aligned}
$$



Allows resummation at NLP with, in principle, arbitrary logarithmic accuracy

But the relation to standard collinear PDFs needs to be further explored:

Threshold PDF depends on the contour

$\mathscr{C} \leftrightarrow$ refactorization scale

ENDPOINT PDF

$$W_\phi = |\,C_g^{A0}\,|^2\, \mathcal{J}^{(g)}(k^2) \otimes \left(f_{g/N}^{\mathrm{thr,LP}}(x) + f_{g/N}^{\mathrm{thr,NLP}}(x, \mathscr{C}) \right)$$

$$+ \sum_{i=q,g} \left[C_{gi}^{B1}(r) C_{gi}^{B1*}(r') \otimes J^{(ig)}(k^2, r, r') \right]_{\mathscr{C}} \otimes f_{i/N}^{\mathrm{thr,LP}} + \mathrm{NNLP}$$

$\overline{\mathrm{MS}}$ standard definition requires calculating convolution integral first and then subtracting the poles

$\boxed{\text{It is incompatible with the factorization paradigm: each single-scale object can be renormalized independently!}}$

Fixed order ✓ | Resummation in d-dimensions ↻ | RGE resummation ✗

SUMMARY AND OUTLOOK

- EFTs allow for a deeper understanding of known problems and systematic computation of unknown sub-leading corrections
- Factorization beyond LP allows the derivation of new valuable results and reveals yet unexplored universal structures:
  - Problems with massive particles have surprisingly complex structure even in QED
  - Amplitude factorization allowed us to reduce theoretical uncertainties
  - PDFs near threshold $\neq$ Threshold expansions of PDFs