Muon $g-2$: data-driven expectations

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FOR FUNDAMENTAL PHYSICS

August 7, 2024

52nd SLAC Summer Institute (SSI 2024)

The Art of Precision: Calculations and Measurements

Lepton dipole moments: the art of precision

- Dipole moments: definition

$$\mathcal{H} = -\boldsymbol{\mu}_\ell \cdot \boldsymbol{B} - \boldsymbol{d}_\ell \cdot \boldsymbol{E}$$

$$\boldsymbol{\mu}_\ell = -g_\ell \frac{e}{2m_\ell}\boldsymbol{S} \qquad \boldsymbol{d}_\ell = -\eta_\ell \frac{e}{2m_\ell}\boldsymbol{S} \qquad a_\ell = \frac{g_\ell - 2}{2}$$

- Anomalous magnetic moments Northwestern 2023, Fermilab 2023

$$a_e^{\text{exp}} = 115{,}965{,}218{,}059(13) \times 10^{-14} \qquad a_\mu^{\text{exp}} = 116{,}592{,}059(22) \times 10^{-11}$$

- Electric dipole moments Roussy et al. 2023, BNL 2009

$$|d_e^{\text{exp}}| < 4.1 \times 10^{-30}\, e\,\text{cm} \qquad |d_\mu^{\text{exp}}| < 1.5 \times 10^{-19}\, e\,\text{cm} \qquad 90\%\ \text{C.L.}$$

- Not much known (yet) about τ dipole moments (in comparison)
- SSI 2024:
  - Muon $g-2$: data-driven expectations this talk
  - Muon $g-2$: experimental status and future talk by James Mott
  - Muon $g-2$: lattice expectations talk by Aida El-Khadra

How to measure the muon $g-2$

- Muon lives long enough to put it into a **storage ring** $\tau_\mu \simeq 2.2\,\mu$s
- Muons produced from pion decay **automatically polarized**
- Frequencies of polarized muons in magnetic field $\boldsymbol{B}$, $\boldsymbol{\beta}\cdot\boldsymbol{B}=0$:
  - **Cyclotron frequency**: $\boldsymbol{\omega}_c = -\frac{q}{m_\mu\gamma}\boldsymbol{B} \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}}$
  - **Spin precession**: $\boldsymbol{\omega}_s = \underbrace{-\frac{g_\mu q}{2m_\mu}\boldsymbol{B}}_{\text{torque of magnetic moment}} \underbrace{-(1-\gamma)\frac{q}{\gamma m_\mu}\boldsymbol{B}}_{\text{Thomas precession for rotating frame}}$
  - **Anomalous precession**: $\boldsymbol{\omega}_a = \boldsymbol{\omega}_s - \boldsymbol{\omega}_c = -\frac{g_\mu-2}{2}\frac{q}{m_\mu}\boldsymbol{B} = -a_\mu\frac{q}{m_\mu}\boldsymbol{B}$
- Including electric field and $\boldsymbol{\beta}\cdot\boldsymbol{B}\neq 0$

$$\boldsymbol{\omega}_a + \boldsymbol{\omega}_{\text{EDM}} = -\frac{q}{m_\mu}\bigg[a_\mu\boldsymbol{B} - a_\mu\frac{\gamma}{\gamma+1}(\boldsymbol{\beta}\cdot\boldsymbol{B})\boldsymbol{\beta} - \bigg(a_\mu - \frac{1}{\gamma^2-1}\bigg)\boldsymbol{\beta}\times\boldsymbol{E} + \frac{\eta}{2}\Big(\boldsymbol{\beta}\times\boldsymbol{B}+\boldsymbol{E}\Big)\bigg]$$

- **"Magic γ"**: $\gamma_{\text{magic}} = \sqrt{1+\frac{1}{a_\mu}} \simeq 29.3 \qquad p_\mu \simeq 3.094\,\text{GeV}$

How to measure the muon $g-2$

BMT equation (Bargmann, Michel, Telegdi 1959)

$$\boldsymbol{\omega}_a + \boldsymbol{\omega}_{\text{EDM}} = -\frac{q}{m_\mu}\left[a_\mu \boldsymbol{B} - a_\mu \frac{\gamma}{\gamma+1}(\boldsymbol{\beta}\cdot\boldsymbol{B})\boldsymbol{\beta} - \left(a_\mu - \frac{1}{\gamma^2-1}\right)\boldsymbol{\beta}\times\boldsymbol{E} + \frac{\eta}{2}\left(\boldsymbol{\beta}\times\boldsymbol{B}+\boldsymbol{E}\right)\right]$$

How to make use of this:

1. Run at **magic γ**: CERN, Brookhaven, Fermilab
   - Various corrections: ***E*-field correction** (imperfect cancellation of $\boldsymbol{\beta}\times\boldsymbol{E}$ term), **pitch correction** (betatron oscillations leading to nonzero average value of $\boldsymbol{\beta}\cdot\boldsymbol{B}$), ...
   - Need highly uniform $\boldsymbol{B}$ field (ppm), detailed field maps with NMR probes
   - Master formula:

$$a_\mu = \frac{\omega_a}{\tilde{\omega}'_p(T_r)}\frac{\mu'_p(T_r)}{\mu_e(H)}\frac{\mu_e(H)}{\mu_e}\frac{m_\mu}{m_e}\frac{g_e}{2}$$

   $\mu'_p(T_r)$: shielded proton magnetic moment at $T_r = 34.7^\circ$C
   - How this is actually done see James's talk

How to measure the muon $g-2$

BMT equation (Bargmann, Michel, Telegdi 1959)

$$\boldsymbol{\omega}_a + \boldsymbol{\omega}_{\text{EDM}} = -\frac{q}{m_\mu}\left[a_\mu \boldsymbol{B} - a_\mu \frac{\gamma}{\gamma+1}(\boldsymbol{\beta}\cdot\boldsymbol{B})\boldsymbol{\beta} - \left(a_\mu - \frac{1}{\gamma^2-1}\right)\boldsymbol{\beta}\times\boldsymbol{E} + \frac{\eta}{2}\left(\boldsymbol{\beta}\times\boldsymbol{B}+\boldsymbol{E}\right)\right]$$

How to make use of this:

2. Run at $\boldsymbol{\beta}\times\boldsymbol{E}=\mathbf{0}$: J-PARC
   - Need **ultracold muons**, negligible transverse momentum
   - γ smaller $\Rightarrow$ lifetime smaller $\Rightarrow$ need higher statistics
3. Cancel $\boldsymbol{B}$ vs. $\boldsymbol{\beta}\times\boldsymbol{E}$ term: **frozen-spin technique**
   - Proposal for dedicated EDM experiment at PSI to improve $|d_\mu|$ by more than three orders of magnitude

How to calculate the muon $g-2$

Vector form factors

$$\langle p'|j^\mu_{\text{em}}|p\rangle = e\bar{u}(p')\Big[\gamma^\mu F_1(s) + \frac{i\sigma^{\mu\nu}q_\nu}{2m_\mu}F_2(s)\Big]u(p) \qquad q = p'-p$$

- **Dirac form factor**: $F_1(0)=1 \Rightarrow$ charge renormalization
- **Pauli form factor**: $F_2(0)=a_\mu$
- In practice, extract $F_2(s)$ via projectors from full vertex function $\Gamma^\mu(p',p)$

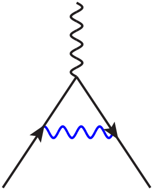
$$F_2(s) = \text{Tr}\Big[(\not{p}+m_\mu)\Lambda_\mu(p,p')(\not{p}'+m_\mu)\Gamma^\mu(p',p)\Big]$$

$$\Lambda_\mu(p,p') = \frac{m_\mu^2}{s(4m_\mu^2-s)}\bigg[\gamma_\mu + \frac{s+2m_\mu^2}{m_\mu(s-4m_\mu^2)}(p+p')_\mu\bigg]$$

How to calculate the muon $g-2$

- Leading order in QED: **Schwinger term**
- Calculate directly for "heavy photon" $\frac{-ig^{\mu\nu}}{k^2-m_\gamma^2+i\epsilon}$

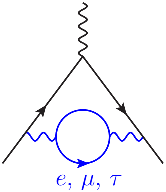
$$a_\mu = \frac{\alpha}{\pi}\int_0^1 dx \frac{x^2(1-x)}{x^2+(1-x)\frac{m_\gamma^2}{m_\mu^2}} \xrightarrow{m_\gamma\to 0} \frac{\alpha}{2\pi}$$



- Neat trick to get lepton loops:
  - Write polarization function as

$$\bar{\Pi}_\ell(q^2) \equiv \Pi_\ell(q^2)-\Pi_\ell(0) = \frac{2\alpha}{\pi}\int_0^1 dx\, x(1-x)\log\frac{m_\ell^2-x(1-x)q^2}{m_\ell^2}$$

$$= \frac{q^2}{\pi}\int_{4m_\ell^2}^\infty ds \frac{\operatorname{Im}\Pi_\ell(s)}{s(s-q^2-i\epsilon)}$$



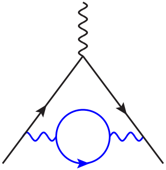
  - Use heavy-photon result above

How to calculate the muon $g-2$

- Neat trick to get lepton loops:
  - Use heavy-photon result above

$$a_\mu^\ell = -\frac{\alpha}{\pi^2}\int_{4m_\ell^2}^\infty ds\,\frac{\text{Im}\,\Pi_\ell(s)}{s}\int_0^1 dx\,\frac{x^2(1-x)}{x^2+(1-x)\frac{s}{m_\mu^2}}$$

$$= \frac{\alpha}{\pi}\int_0^1 dx(1-x)\bar{\Pi}_\ell\bigg(-\frac{x^2m_\mu^2}{1-x}\bigg)$$

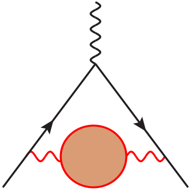


  - Reproduces

$$a_\mu^e = \Big(\frac{\alpha}{\pi}\Big)^2\bigg[\frac{1}{3}\log\frac{m_\mu}{m_e}-\frac{25}{36}+\ldots\bigg]$$

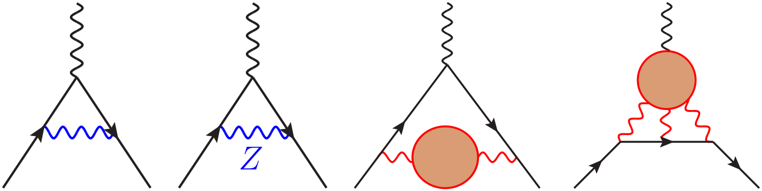
$$a_\mu^\mu = \Big(\frac{\alpha}{\pi}\Big)^2\bigg[\frac{119}{36}-\frac{\pi^2}{3}\bigg] \qquad a_\mu^\tau = \frac{1}{45}\Big(\frac{m_\mu}{m_\tau}\Big)^2\Big(\frac{\alpha}{\pi}\Big)^2+\ldots$$

- Same idea works if only $\text{Im}\,\Pi(s)$ is known
  ↪ hadronic contributions



$$\text{Im}\,\Pi_{\text{had}}(s) = -\frac{s}{4\pi\alpha}\sigma_{\text{tot}}(e^+e^-\to\text{hadrons})$$

Anomalous magnetic moments of charged leptons



- **SM prediction for $(g-2)_\ell$**

$$a_\ell^{\text{SM}} = a_\ell^{\text{QED}} + a_\ell^{\text{EW}} + a_\ell^{\text{had}}$$

- For the electron: electroweak and hadronic contributions under control
- For a precision calculation need:
  - Independent input for α
  - Higher-order QED contributions
- For the muon: by far main uncertainty from the **hadronic contributions**

  $\hookrightarrow$ focus of this lecture

SM prediction for $(g-2)_\mu$: QED

- **5-loop QED** result Aoyama, Kinoshita, Nio 2018:

$$a_\mu^{\text{QED}} = 116\,584\,719.0(1) \times 10^{-11}$$

$\hookrightarrow$ insensitive to input for α (at this level)

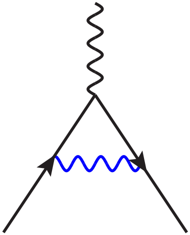
- QED coefficients enhanced by $\log m_\mu / m_e$
- Enhancement from naive RG expectation for 6-loop QED

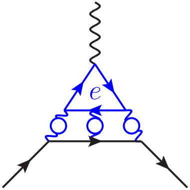
$$10 \times \frac{2}{3}\pi^2 \log \frac{m_\mu}{m_e} \times \left(\frac{2}{3} \log \frac{m_\mu}{m_e}\right)^3 \simeq 1.6 \times 10^4$$

$\hookrightarrow$ would imply $a_\mu^{\text{6-loop}} \simeq 0.2 \times 10^{-11}$

- Refined RG estimate Aoyama, Hayakawa, Kinoshita, Nio 2012

$$a_\mu^{\text{6-loop}} \simeq 0.1 \times 10^{-11}$$





SM prediction for $(g-2)_\mu$: electroweak

- Electroweak contribution Gnendiger et al. 2013

$$a_\mu^{\text{EW}} = (194.8 - 41.2) \times 10^{-11} = 153.6(1.0) \times 10^{-11}$$

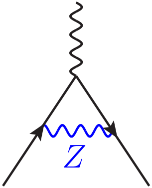
- Remaining uncertainty dominated by $q = u, d, s$ loops

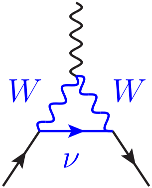
  $\hookrightarrow$ nonperturbative effects Czarnecki, Marciano, Vainshtein 2003

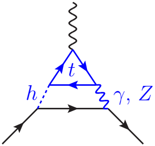
- First time data-driven methods enter

  $\hookrightarrow$ **hadronic VVA correlator**

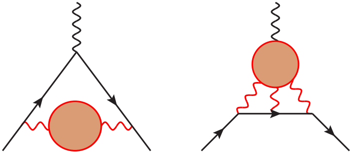
- 3-loop corrections?
  - 3-loop RG estimate accidentally cancels in scheme chosen by Gnendiger et al. 2013, with an (NLL) error of 0.2×10^{-11}
  - α_s corrections for heavy quarks Melnikov 2006







SM prediction for $(g-2)_\mu$: hadronic effects



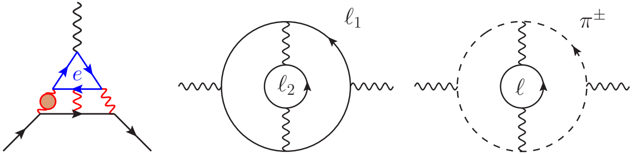
- **Hadronic vacuum polarization**: need hadronic two-point function

$$\Pi_{\mu\nu} = \langle 0|T\{j_\mu j_\nu\}|0\rangle$$

- **Hadronic light-by-light scattering**: need hadronic four-point function

$$\Pi_{\mu\nu\lambda\sigma} = \langle 0|T\{j_\mu j_\nu j_\lambda j_\sigma\}|0\rangle$$

SM prediction for $(g-2)_\mu$: higher-order hadronic effects



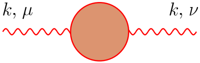
- Generic scaling of $\mathcal{O}(\alpha^4)$ effects: $\left(\frac{\alpha}{\pi}\right)^4 \simeq 3 \times 10^{-11}$
- Enhancements (numerical or $\log \frac{m_e}{m_\mu}$) can make such effects relevant

  $\hookrightarrow$ NNLO HVP iterations need to be included Kurz et al. 2014
- NLO HLbL small Colangelo et al. 2014
- Mixed hadronic and leptonic contributions with inner electron potentially dangerous

  $\hookrightarrow$ could affect LO HVP via radiation of e^+e^- pairs, but $\lesssim 1 \times 10^{-11}$ MH, Teubner 2022

Hadronic vacuum polarization

- General principles yield **direct connection with experiment**
  - **Gauge invariance**



$$= -i\big(k^2 g^{\mu\nu} - k^\mu k^\nu\big)\Pi(k^2)$$

  - **Analyticity**

$$\Pi_{\text{ren}} = \Pi(k^2) - \Pi(0) = \frac{k^2}{\pi}\int_{4M_\pi^2}^{\infty} \text{d}s \frac{\text{Im}\,\Pi(s)}{s(s-k^2)}$$

  - **Unitarity**

$$\text{Im}\,\Pi(s) = -\frac{s}{4\pi\alpha}\sigma_{\text{tot}}\big(e^+e^- \to \text{hadrons}\big) = -\frac{\alpha}{3}R_{\text{had}}(s)$$

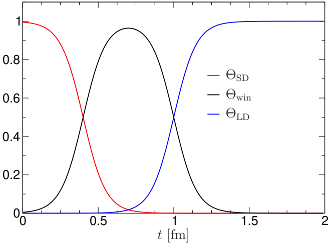
- Resulting master formula Bouchiat, Michel 1961, Brodsky, de Rafael, 1968

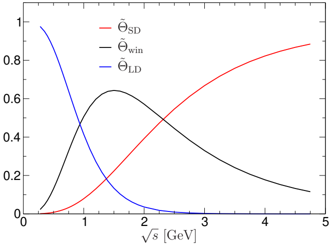
$$a_\mu^{\text{HVP, LO}} = \left(\frac{\alpha m_\mu}{3\pi}\right)^2 \int_{s_{\text{thr}}}^{\infty} ds \frac{\hat{K}(s)}{s^2} R_{\text{had}}(s) \qquad R_{\text{had}}(s) = \frac{3s}{4\pi\alpha^2}\sigma_{\text{tot}}(e^+e^- \to \text{hadrons}(+\gamma))$$

- Main challenge: measure hadronic cross sections at better than 1% precision

  ↪ **radiative corrections**

Hadronic vacuum polarization: windows in Euclidean time





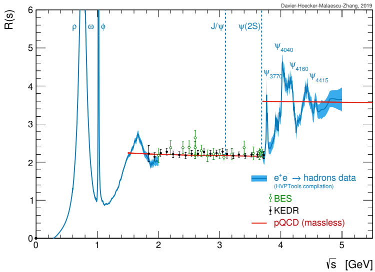
- Idea RBC/UKQCD 2018: define partial quantities (**Euclidean windows**)

$$a_\mu^{\text{HVP, LO, win}} = \left(\frac{\alpha m_\mu}{3\pi}\right)^2 \int_{s_{\text{thr}}}^\infty ds \frac{\hat{K}(s)}{s^2} R_{\text{had}}(s)\tilde{\Theta}_{\text{win}}(s)$$

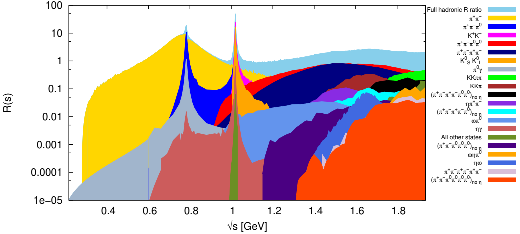
$\hookrightarrow$ smaller systematic errors for same quantity in lattice QCD see Aida's talk

- Separation of full HVP into
  - **Long-distance** window (LD): 1 fm $\lesssim t$ $\Rightarrow$ $a_\mu^{\text{HVP, LO, LD}} \simeq 57\%$
  - **Intermediate** window (win): 0.4 fm $\lesssim t \lesssim$ 1 fm $\Rightarrow$ $a_\mu^{\text{HVP, LO, win}} \simeq 33\%$
  - **Short-distance** window (SD): $t \lesssim$ 0.4 fm $\Rightarrow$ $a_\mu^{\text{HVP, LO, SD}} \simeq 10\%$

Hadronic vacuum polarization from e^+e^- data



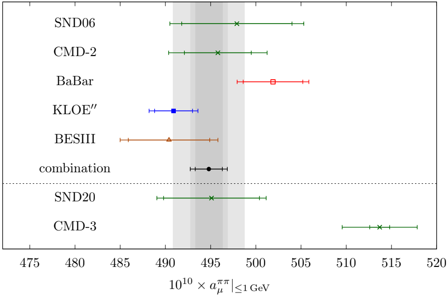
Davier, Hoecker, Malaescu, Zhang 2019



Keshavarzi, Nomura, Teubner 2018

- Decades-long effort to measure e^+e^- cross sections
  - cross sections defined photon-inclusively
    $\hookrightarrow$ threshold $s_{\text{thr}} = M_{\pi^0}^2$ due to $\pi^0\gamma$ channel
  - up to about 2 GeV: sum of exclusive channels
  - above: inclusive data + narrow resonances + pQCD
- **Tensions in the data**: long-standing one between KLOE and BaBar 2π data, became much worse with CMD-3

The current picture for $e^+e^- \to \pi^+\pi^-$



	$a_\mu^{\pi\pi}\|_{\leq 1\,\text{GeV}}$	$a_\mu^{\pi\pi}\|_{[0.60,0.88]\,\text{GeV}}$	$a_\mu^{\pi\pi}\|_{\text{win}}$
SND06	1.7σ	1.8σ	1.7σ
CMD-2	2.0σ	2.3σ	2.1σ
BaBar	2.9σ	3.3σ	3.1σ
KLOE′′	4.8σ	5.6σ	5.4σ
BESIII	2.8σ	3.0σ	3.1σ
SND20	2.1σ	2.2σ	2.2σ
comb	3.7σ [5.0σ]	4.2σ [6.1σ]	3.8σ [5.7σ]

- CMD-3 disagrees with previous measurements at the level of $(2–5)\sigma$
- But: the resulting picture agrees well with the one emerging from recent lattice results BMWc 24, RBC/UKQCD 24, see Aida's talk
- Now what?
  - New 2π measurements forthcoming: BaBar, KLOE, SND, BES III, Belle II
  - Need to understand origin of differences: radiative corrections, MC generators

Analyticity constraints on $e^+e^- \to$ hadrons cross sections

- HVP integral dominated by a few channels for which high precision is required

  $\hookrightarrow$ $e^+e^- \to \pi^+\pi^-, 3\pi, \bar{K}K, \ldots$
- These channels are determined by (reasonably) simple matrix elements
  - $\pi^+\pi^-, \bar{K}K$: electromagnetic form factor
  - 3π: matrix element for $\gamma^* \to 3\pi$

  $\hookrightarrow$ for these objects **further constraints from analyticity and unitarity** apply!
- Why bother, since anyway cross sections are measured?
  - **Cross checks** on data sets

    $\hookrightarrow$ need to comply with QCD constraints
  - Improve **precision**, evaluate over entire kinematic range see 2π plot above
  - **Correlations** with other low-energy observables
  - **Structure-dependent radiative corrections**
  - Understand **anatomy** of cross sections

    $\hookrightarrow$ comparison with lattice QCD see Aida's talk

Dispersive representation of the pion form factor

The pion form factor from dispersion relations

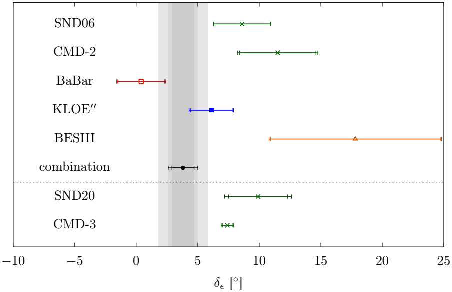
$$F_\pi^V(s) = \underbrace{\Omega_1^1(s)}_{\text{elastic } \pi\pi \text{ scattering}} \times \underbrace{G_\omega(s)}_{\text{isospin-breaking } 3\pi \text{ cut}} \times \underbrace{G_{\text{in}}(s)}_{\text{inelastic effects: } 4\pi, \ldots}$$

$$\Omega_1^1(s) = \exp\left\{\frac{s}{\pi}\int_{4M_\pi^2}^\infty ds' \frac{\delta_1^1(s')}{s'(s'-s)}\right\} \qquad G_\omega(s) \simeq 1 + \frac{s\epsilon_\omega}{M_\omega^2 - s - iM_\omega\Gamma_\omega}$$

- $e^+e^- \to \pi^+\pi^-$ cross section subject to strong constraints from **analyticity**, **unitarity**, **crossing symmetry**, leading to dispersive representation with few parameters Colangelo, MH, Stoffer, 2018, 2021, 2022, work in progress
  - **Elastic $\pi\pi$ scattering**: two values of phase shifts
  - **ρ–ω mixing**: ω pole parameters and residue
  - **Inelastic states**: conformal polynomial

$\hookrightarrow$ correlations with $\pi\pi$ phase shifts, pion charge radius, . . .

Phase of the ρ–ω mixing parameter



- Can also study consistency of hadronic parameters
  $\hookrightarrow$ **phase of the ρ–ω mixing parameter δ_ϵ**
  - δ_ϵ observable, since defined as a phase of a residue
  - δ_ϵ vanishes in isospin limit, but can be non-vanishing due to $\rho \to \pi^0\gamma, \eta\gamma, \pi\pi\gamma, \ldots \to \omega$
  - Combined-fit $\delta_\epsilon = 3.8(2.0)[1.2]^\circ$ agrees well with narrow-width expectation $\delta_\epsilon = 3.5(1.0)^\circ$, but **considerable spread among experiments**
  - Mass of the ω systematically too low compared to $e^+e^- \to 3\pi$

Radiative corrections and MC generators

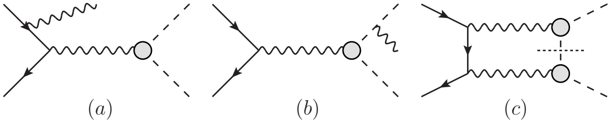
- How to evaluate radiative corrections for processes involving hadrons?
- Ongoing comparative study of MC generators STRONG2020
- Two classes of experiments:
  - **Energy scan**: CMD-3, SND
  - **Initial state radiation**: KLOE, BaBar, BES III, Belle II
- So far for $\pi^+\pi^-$: based on **scalar QED (point-like pions)**
- $F \times$ sQED: pion form factors included Campanario et al. 2019

  $\hookrightarrow$ either $F_\pi^V(s)$ (e^+e^- invariant mass) or $F_\pi^V(q^2)$ ($\pi^+\pi^-$ invariant mass)
- Captures correctly all the infrared properties
- Potential issues:
  - **Structure-dependent corrections** CMD-3

    $\hookrightarrow$ $F \times$ sQED might not be sufficient for ISR experiments
  - **Multiple photon emission** BaBar 2023

    $\hookrightarrow$ effects can be enhanced by experimental cuts

Radiative corrections: forward–backward asymmetry

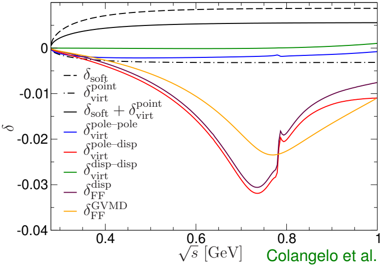


- Consider **forward–backward asymmetry** $A_{\text{FB}}(z) = \frac{\frac{d\sigma}{dz}(z) - \frac{d\sigma}{dz}(-z)}{\frac{d\sigma}{dz}(z) + \frac{d\sigma}{dz}(-z)}$ for energy scan
  $\hookrightarrow$ ***C*-odd**, only generated at loop level
- CMD-3 observed that $F \times$ sQED fails for diagram (c), use generalized vector meson dominance instead Ignatov, Lee 2022
- Problem: unphysical imaginary parts below 2π threshold in loop integral
- Better approach: use **dispersive representation of pion VFF**

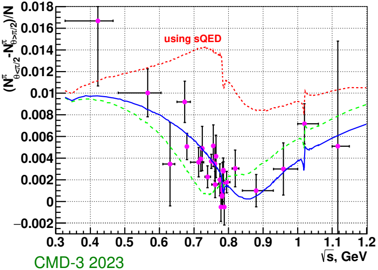
$$\frac{F_\pi^V(s)}{s} = \frac{1}{s} + \frac{1}{\pi}\int_{4M_\pi^2}^\infty ds' \frac{\operatorname{Im} F_\pi^V(s')}{s'(s'-s)} \to \frac{1}{s-\lambda^2} - \frac{1}{\pi}\int_{4M_\pi^2}^\infty ds' \frac{\operatorname{Im} F_\pi^V(s')}{s'} \frac{1}{s-s'}$$

$\hookrightarrow$ captures all the **structure-dependent, infrared-enhanced effects**

Radiative corrections: forward–backward asymmetry



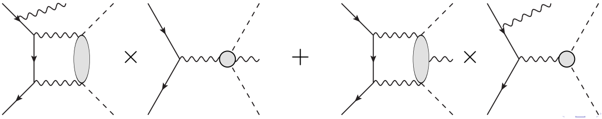
Colangelo et al. 2022



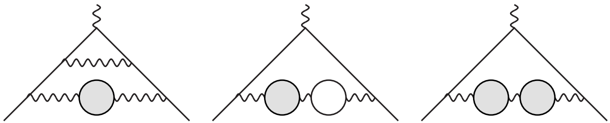
CMD-3 2023

- Reasonable agreement between dispersive formulation and GVMD!
- Are there relevant effects being missed in the *C*-even contributions?

  $\hookrightarrow$ potentially relevant for ISR experiments Ignatov, STRONG2020
- ISR–FSR interference:



Do e^+e^- data and lattice really measure the same thing?



- Conventions for **bare cross section**
  - Includes radiative intermediate states and final-state radiation: $\pi^0\gamma$, $\eta\gamma$, $\pi\pi\gamma$, $\ldots$
  - Initial-state radiation and VP subtracted to avoid double counting
- NLO HVP insertions

$$a_\mu^{\text{HVP, NLO}} \simeq [\underbrace{-20.7}_{(a)} + \underbrace{10.6}_{(b)} + \underbrace{0.3}_{(c)}] \times 10^{-10} = -9.8 \times 10^{-10}$$

$\hookrightarrow$ dominant VP effect from leptons, HVP iteration very small

- Important point: **no need to specify hadronic resonances**

$\hookrightarrow$ calculation set up in terms of decay channels

Do e^+e^- data and lattice really measure the same thing?

- HVP in subtraction determined iteratively (converges with α) and self-consistently

$$\alpha(q^2) = \frac{\alpha(0)}{1 - \Delta\alpha_{\text{lep}}(q^2) - \Delta\alpha_{\text{had}}(q^2)} \qquad \Delta\alpha_{\text{had}}(q^2) = -\frac{\alpha q^2}{3\pi} P \int_{s_{\text{thr}}}^{\infty} ds \frac{R_{\text{had}}(s)}{s(s-q^2)}$$

- Subtlety for very narrow $c\bar{c}$ and $b\bar{b}$ resonances (ω and ϕ perfectly fine)

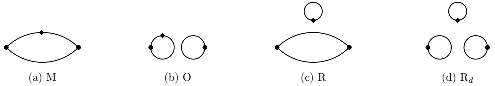
  $\hookrightarrow$ Dyson series does not converge Jegerlehner

- Solution: take out resonance that is being corrected in R_{had} in VP undressing
- How to match all of this on the lattice?
- Need to calculate all sorts of **isospin-breaking (IB) corrections**

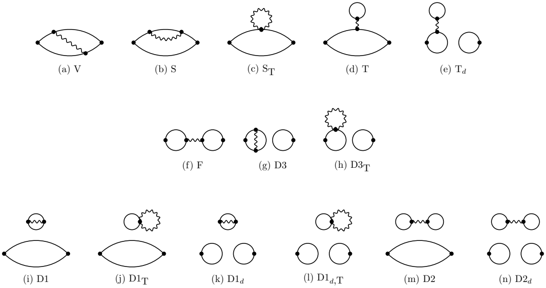
  $\hookrightarrow$ e^2 (QED) and $\delta = m_u - m_d$ (strong IB) corrections

Do e^+e^- data and lattice really measure the same thing?

- **Strong isospin breaking** $\propto m_u - m_d$



- **QED effects** $\propto \alpha$



plots from Gülpers et al. 2018

- Diagram (f) F critical for consistent VP subtraction

  ↪ same diagram without additional gluons is subtracted RBC/UKQCD 2018

Estimating isospin-breaking effects from phenomenology

	SD window		int window		LD window		full HVP	
	$\mathcal{O}(e^2)$	$\mathcal{O}(\delta)$	$\mathcal{O}(e^2)$	$\mathcal{O}(\delta)$	$\mathcal{O}(e^2)$	$\mathcal{O}(\delta)$	$\mathcal{O}(e^2)$	$\mathcal{O}(\delta)$
$\pi^0\gamma$	0.16(0)	–	1.52(2)	–	2.70(4)	–	4.38(6)	–
$\eta\gamma$	0.05(0)	–	0.34(1)	–	0.31(1)	–	0.70(2)	–
$\omega(\to\pi^0\gamma)\pi^0$	0.15(0)	–	0.54(1)	–	0.19(0)	–	0.88(2)	–
FSR (2π)	0.12(0)	–	1.17(1)	–	3.13(3)	–	4.42(4)	–
FSR (3π)	0.03(0)	–	0.20(0)	–	0.28(1)	–	0.51(1)	–
FSR (K^+K^-)	0.07(0)	–	0.39(2)	–	0.29(2)	–	0.75(4)	–
ρ–ω mixing (2π)	–	0.06(1)	–	0.86(6)	–	2.87(12)	–	3.79(19)
ρ–ω mixing (3π)	–	−0.13(3)	–	−1.03(27)	–	−1.52(40)	–	−2.68(70)
pion mass (2π)	0.04(8)	–	−0.09(56)	–	−7.62(63)	–	−7.67(94)	–
kaon mass (K^+K^-)	−0.29(1)	0.44(2)	−1.71(9)	2.63(14)	−1.24(6)	1.91(10)	−3.24(17)	4.98(26)
kaon mass $(\bar{K}^0K^0)$	0.00(0)	−0.41(2)	−0.01(0)	−2.44(12)	−0.01(0)	−1.78(9)	−0.02(0)	−4.62(23)
sum	0.33(8)	−0.04(4)	2.34(57)	0.02(33)	−1.97(63)	1.48(44)	0.71(95)	1.47(80)

MH et al. 2023

$\hookrightarrow$ **individually sizable results** that largely cancel in the end

Estimating isospin-breaking effects from phenomenology

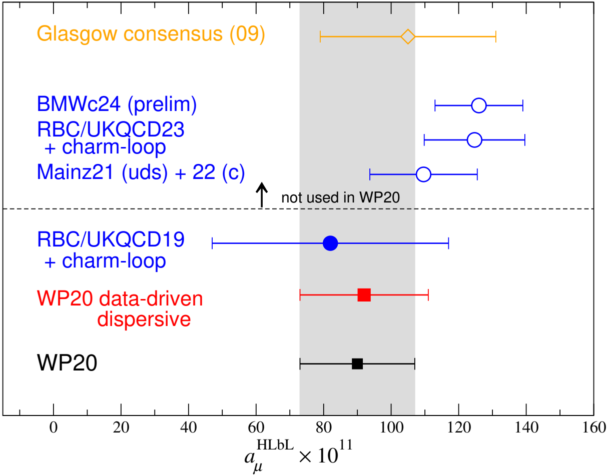
		this work	BMWc 2020	RBC/UKQCD 2018
SD	$\mathcal{O}(e^2)$	0.33(8)(8)(49)[51]	–	–
	$\mathcal{O}(\delta)$	$-0.04(4)(8)(49)[50]$	–	–
int	$\mathcal{O}(e^2)$	2.34(57)(47)(55)[92]	$-0.09(6)$	0.0(2)
	$\mathcal{O}(\delta)$	0.02(33)(47)(55)[79]	0.52(4)	0.1(3)
LD	$\mathcal{O}(e^2)$	$-1.97(63)(36)(12)[74]$	–	–
	$\mathcal{O}(\delta)$	1.48(44)(36)(12)[58]	–	–
full	$\mathcal{O}(e^2)$	0.71(0.95)(0.90)(1.16)[1.75]	$-1.5(6)$	$-1.0(6.6)$
	$\mathcal{O}(\delta)$	1.47(0.80)(0.90)(1.16)[1.67]	1.9(1.2)	10.6(8.0)

- Reasonable agreement with BMWc 2020, RBC/UKQCD 2018

  $\hookrightarrow$ if anything, the result would become even larger with pheno estimates

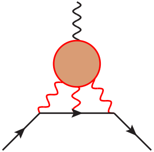
- Isospin-breaking contributions are very unlikely to be the reason for the lattice vs. phenomenology tension

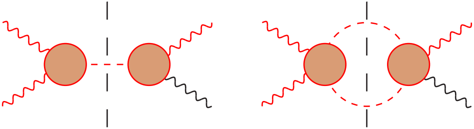
HLbL scattering: status



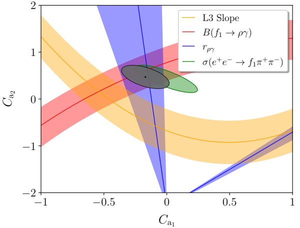
- Good agreement between lattice QCD and phenomenology at $\simeq 20 \times 10^{-11}$
- Need another factor of 2 for final Fermilab precision see James's talk

HLbL scattering: data-driven, dispersive evaluations

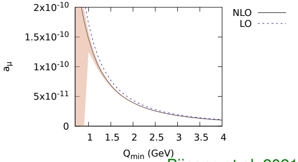




- Organized in terms of **hadronic intermediate states**, in close analogy to HVP Colangelo et al. 2014, ...
- Leading channels implemented with **data input for $\gamma^*\gamma^* \to$ hadrons**, e.g., $\pi^0 \to \gamma^*\gamma^*$
- Uncertainty dominated by subleading channels
  $\hookrightarrow$ **axial-vector mesons** $f_1(1285)$, $f_1(1420)$, $a_1(1260)$
- Optimized HLbL basis MH, Stoffer, Zillinger 2024
- Matching to short-distance constraints



MH, Kubis, Zanke 2023

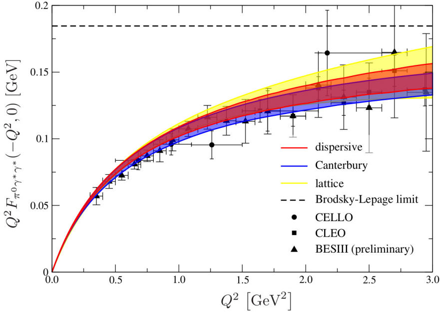


Bijnens et al. 2021

HLbL scattering: white paper details

Contribution	PdRV(09)	N/JN(09)	J(17)	Our estimate
π^0, η, η'-poles	114(13)	99(16)	95.45(12.40)	93.8(4.0)
π, K-loops/boxes	$-19(19)$	$-19(13)$	$-20(5)$	$-16.4(2)$
S-wave $\pi\pi$ rescattering	$-7(7)$	$-7(2)$	$-5.98(1.20)$	$-8(1)$
subtotal	88(24)	73(21)	69.5(13.4)	69.4(4.1)
scalars	–	–	–	} $-1(3)$
tensors	–	–	1.1(1)	
axial vectors	15(10)	22(5)	7.55(2.71)	6(6)
u, d, s-loops / short-distance	–	21(3)	20(4)	15(10)
c-loop	2.3	–	2.3(2)	3(1)
total	105(26)	116(39)	100.4(28.2)	92(19)

HLbL scattering: pseudoscalar poles



- Pion pole from data MH et al. 2018, Masjuan, Sánchez-Puertas 2017 and lattice Gérardin et al. 2019

$$a_\mu^{\pi^0\text{-pole}}\Big|_{\text{dispersive}} = 63.0^{+2.7}_{-2.1} \times 10^{-11} \qquad a_\mu^{\pi^0\text{-pole}}\Big|_{\text{Canterbury}} = 63.6(2.7) \times 10^{-11}$$

$$a_\mu^{\pi^0\text{-pole}}\Big|_{\text{lattice+PrimEx}} = 62.3(2.3) \times 10^{-11} \qquad a_\mu^{\pi^0\text{-pole}}\Big|_{\text{lattice}} = 59.7(3.6) \times 10^{-11}$$

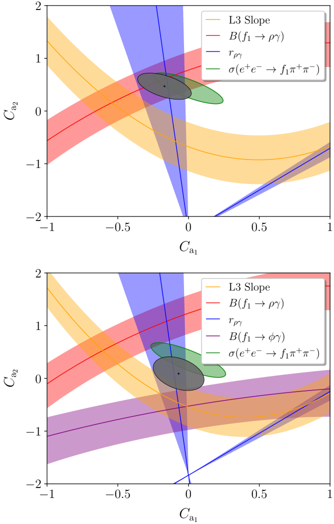
- Singly-virtual results agree well with BESIII measurement
- Same program in progress for η, η' poles
- New lattice results indicate some tension in $\gamma\gamma$ width

Determination of axial-vector TFFs

- Three independent TFFs, accessible in
  - $e^+e^- \to e^+e^- f_1$ (space-like)
  - $f_1 \to \rho\gamma$, $f_1 \to \phi\gamma$
  - $f_1 \to e^+e^-$
  - $\boldsymbol{e^+e^- \to f_1\pi^+\pi^-}$

  $\hookrightarrow$ global analysis in VMD parameterizations
- Constraint from $e^+e^- \to f_1\pi^+\pi^-$ for the first time allows for unambiguous solutions
- Most information available for f_1

  $\hookrightarrow$ f_1' and a_1 from $U(3)$ symmetry
- Analysis of consequences for HLbL in progress



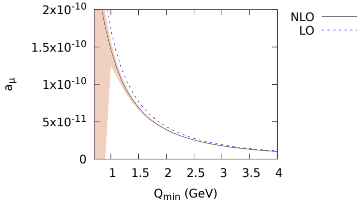
MH, Kubis, Zanke 2023

Short-distance contributions

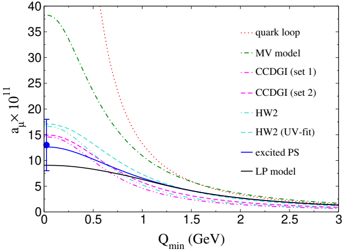
- **Higher-order short-distance constraints**
  - Two-loop α_s corrections
  - Higher-order OPE corrections
  - Higher-order terms in Melnikov–Vainshtein limit
- **Implementation of SDCs**
  - Large-N_c Regge models Colangelo ...
  - Holographic QCD Leutgeb, Rebhan, Cappiello, ...
  - Interpolants Lüdtke, Procura

$\hookrightarrow$ reasonable agreement on longitudinal component

- **Transverse component/axial-vectors**
  - SDCs MH, Stoffer 2020
  - Implementation of axial-vectors, new HLbL basis, new dispersive formalism



Bijnens, Hermansson-Truedsson, Laub, Rodríguez-Sánchez 2021



Colangelo, Hagelstein, MH, Laub, Stoffer 2021

New insights on HLbL tensor

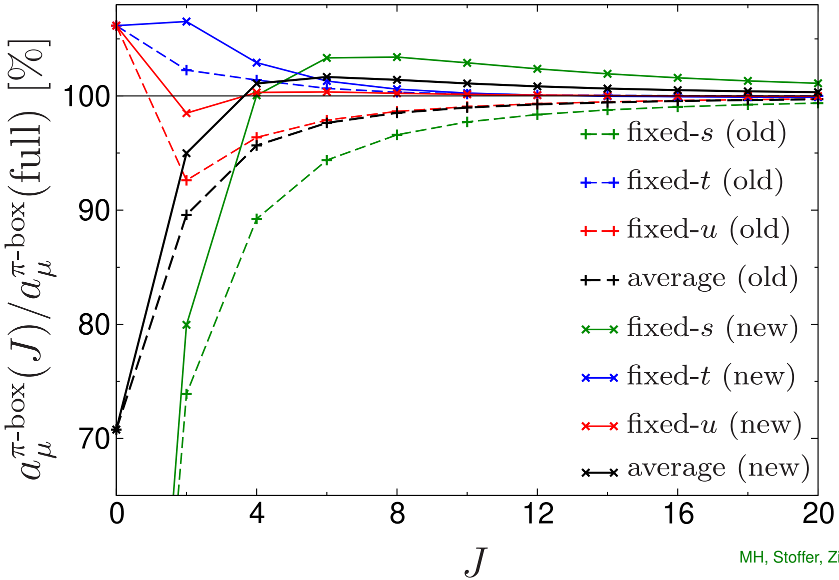
- Recall discussions with MV about the **definition of the pion pole**

$$\frac{F_{\pi^0\gamma^*\gamma^*}(q_1^2,q_2^2)F_{\pi^0\gamma^*\gamma^*}(q_3^2,0)}{q_3^2-M_\pi^2} \quad \text{vs.} \quad \frac{F_{\pi^0\gamma^*\gamma^*}(q_1^2,q_2^2)F_{\pi^0\gamma^*\gamma^*}(M_\pi^2,0)}{q_3^2-M_\pi^2}$$

- Comparison in Colangelo, Hagelstein, MH, Laub, Stoffer 2019:
  - First variant: dispersion relation in four-point kinematics
  - Second variant: dispersion relation in $g-2$ ("triangle") kinematics
- Triangle variant looks attractive because of SDCs, but very complicated in low-energy region due to missing $2\pi, \ldots$ cuts
- **Kinematic singularities**
  - Disappear in four-point kinematics only for the entire HLbL tensor due to sum rules
    $\hookrightarrow$ higher partial waves, axial-vectors, tensors
  - For axial-vectors: can find a basis manifestly free of kinematic singularities
    $\hookrightarrow$ ideal for axial-vectors, also good for pion box; not possible for tensors

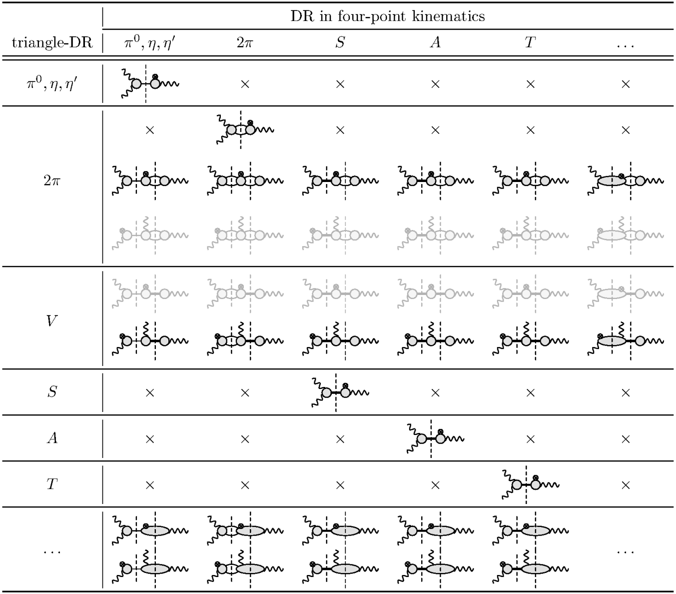
$\hookrightarrow$ complementary information from triangle kinematics Lüdtke, Procura, Stoffer 2023

Saturation of the pion box in new basis



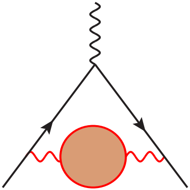
MH, Stoffer, Zillinger 2024

HLbL dispersion relation in triangle vs. four-point kinematics

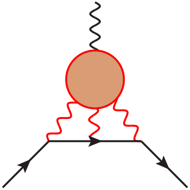


Lüdtke, Procura, Stoffer 2023

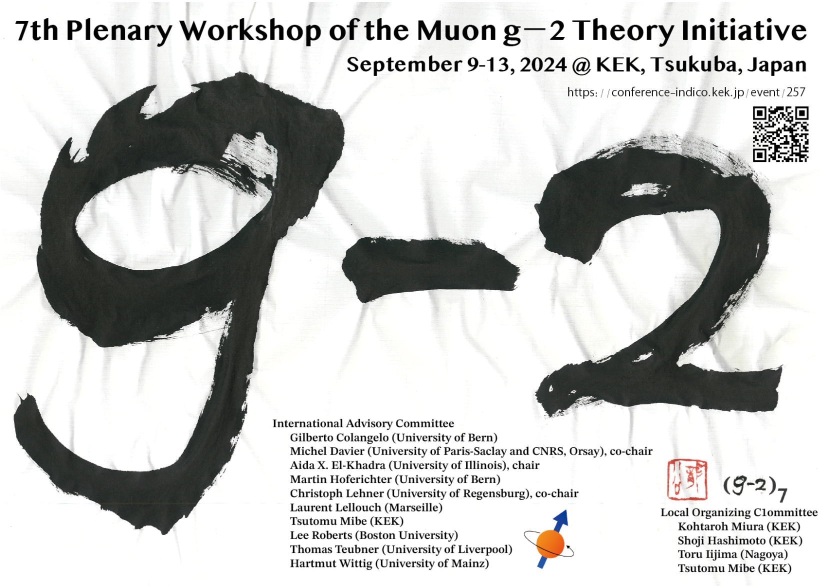
- **Hadronic vacuum polarization**
  - By far largest systematic uncertainty in $\pi\pi$ channel
  - Large range from KLOE to CMD-3, well beyond the quoted errors
  - New data to come: BaBar, KLOE, SND, BES III, Belle II
  - Intense scrutiny of radiative corrections and MC generators



- **Hadronic light-by-light scattering**
  - Use dispersion relations to remove model dependence as far as possible
  - Implemented for leading intermediate states
  - Subleading terms including asymptotic constraints in progress
  - Good agreement between phenomenology and lattice



Seventh plenary workshop of the Muon $g-2$ Theory Initiative



Anomalous magnetic moment of the electron: fine-structure constant

- Input from **atom interferometry**

$$\alpha^2 = \frac{4\pi R_\infty}{c} \times \frac{m_{\text{atom}}}{m_e} \times \frac{\hbar}{m_{\text{atom}}}$$

- With **Rb measurement** LKB 2011 (a_e^{exp} Harvard 2008)

$$a_e^{\text{exp}} = 1{,}159{,}652{,}180.73(28) \times 10^{-12}$$

$$a_e^{\text{SM}} = 1{,}159{,}652{,}182.03(1)_{\text{5-loop}}(1)_{\text{had}}(72)_{\alpha(\text{Rb})} \times 10^{-12}$$

$$a_e^{\text{exp}} - a_e^{\text{SM}} = -1.30(77) \times 10^{-12}[1.7\sigma]$$

$\hookrightarrow$ α limiting factor, but more than an order of magnitude to go in theory

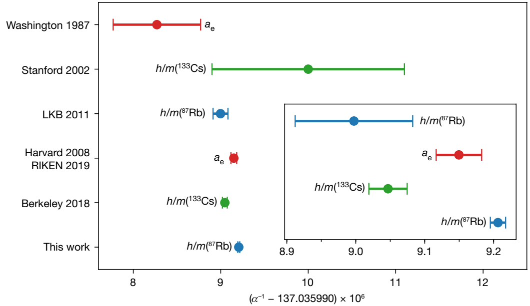
- With **Cs measurement** Berkeley 2018, Science 360 (2018) 191

$$a_e^{\text{SM}} = 1{,}159{,}652{,}181.61(1)_{\text{5-loop}}(1)_{\text{had}}(23)_{\alpha(\text{Cs})} \times 10^{-12}$$

$$a_e^{\text{exp}} - a_e^{\text{SM}} = -0.88(36) \times 10^{-12}[2.5\sigma]$$

$\hookrightarrow$ for the first time a_e^{exp} limiting factor

Anomalous magnetic moment of the electron: fine-structure constant



LKB 2020

During the interferometer sequence, we apply a frequency ramp to compensate the Doppler shift induced by gravity. Nonlinearity in the delay of the optical phase-lock loop induces a residual phase shift that is measured and corrected for each spectrum. These systematic effects were not considered in our previous measurement[18] (see Fig. 1), which could explain the 2.4σ discrepancy between that measurement and the present one. Unfortunately, we do not have available data to evaluate retrospectively the contributions of the phase shift in the Raman phase-lock loop and of short-scale fluctuations in the laser intensity to the 2011 measurement. Thus, we cannot firmly state that these two effects are the cause of the 2.4σ discrepancy between our two measurements.

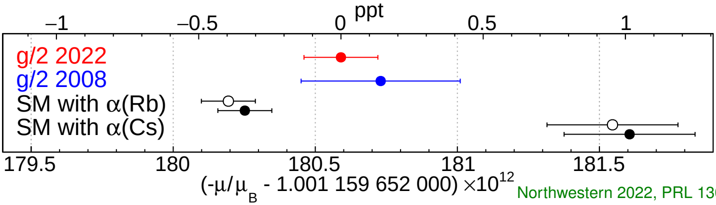
- Tensions
  - Berkeley 2018 vs. LKB 2020: 5.4σ
  - LKB 2011 vs. LKB 2020: 2.4σ

- With new **Rb measurement** LKB 2020, Nature 588 (2020) 61

$$a_e^{\mathrm{SM}} = 1{,}159{,}652{,}180.25(1)_{\text{5-loop}}(1)_{\text{had}}(9)_{\alpha(\mathrm{Rb})} \times 10^{-12}$$

$$a_e^{\mathrm{exp}} - a_e^{\mathrm{SM}} = 0.48(30) \times 10^{-12} [1.6\sigma]$$

Anomalous magnetic moment of the electron: fine-structure constant



Northwestern 2022, PRL 130 (2023) 071801

- Latest development: new measurement of a_e^{exp}

$$a_e^{\text{exp}} = 1{,}159{,}652{,}180.59(13) \times 10^{-12}$$
$$a_e^{\text{exp}} - a_e^{\text{SM}}[\text{Cs}] = -1.02(26) \times 10^{-12}[3.9\sigma]$$
$$a_e^{\text{exp}} - a_e^{\text{SM}}[\text{Rb}] = 0.34(16) \times 10^{-12}[2.1\sigma]$$

- Another 4.8σ tension in 5-loop QED coefficient

  ↪ full circles Aoyama et al. 2019 vs. open circles Volkov 2019

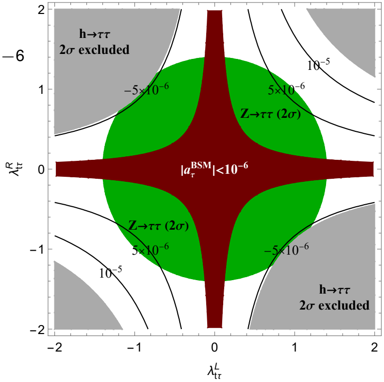
- BSM sensitivity of a_e depends on resolution of this experimental 5σ discrepancy!

What about $(g-2)_\tau$?

- Current status Abdallah et al. 2004, Keshavarzi et al. 2020

$$a_\tau^{\text{exp}} = -0.018(17) \quad \textit{vs.} \quad a_\tau^{\text{SM}} = 1{,}177.171(39)\times 10^{-6}$$

- **Scaling arguments**:
  - Minimal flavor violation: $a_\tau^{\text{BSM}} \simeq a_\mu^{\text{BSM}}\left(\frac{m_\tau}{m_\mu}\right)^2 \simeq 0.7\times 10^{-6}$
  - Electroweak contribution: $a_\tau^{\text{EW}} \simeq 0.5\times 10^{-6}$
- **Concrete models**:
  - S_1 leptoquark model promising due to **chiral enhancement** with $\frac{m_t}{m_\tau}$ $\hookrightarrow$ can get $a_\tau^{\text{BSM}} \simeq (\text{few})\times 10^{-6}$ without violating $h\to\tau\tau$ and $Z\to\tau\tau$



Crivellin, MH, Roney 2021

- Ultimate target has to be a measurement of a_τ at the level of 10^{-6}

$\hookrightarrow$ requires two-loop accuracy for theory throughout

Experimental prospects for $(g-2)_\tau$

- Many recent proposals, none of which seem to reach much beyond the Schwinger term
- Exception: **$e^+e^- \to \tau^+\tau^-$ at Υ resonances** Bernabéu et al. 2007
  $\hookrightarrow$ quotes projections at 10^{-6} level
- Idea: study $e^+e^- \to \tau^+\tau^-$ cross section and asymmetries
  $\hookrightarrow$ could this be realized at Belle II Crivellin, MH, Roney 2021?
- Answer: yes, but requires **polarization upgrade of SuperKEK** to get access to transverse and longitudinal asymmetries
- Idea: extract $F_2(s)$ at $s \simeq (10\,\text{GeV})^2$, but heavy new physics decouples
  $\hookrightarrow$ $a_\tau^{\text{BSM}} = F_2^{\text{exp}}(s) - F_2^{\text{SM}}(s)$ as long as $s \ll \Lambda^2_{\text{BSM}}$
- Bounds on light BSM become model dependent, but anyway better constrained in other processes

First attempt: total cross section

- **Differential cross section** for $e^+e^- \to \tau^+\tau^-$

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2\beta}{4s}\left[(2-\beta^2\sin^2\theta)\Big(|F_1|^2-\gamma^2|F_2|^2\Big)+4\text{Re}\,\big(F_1F_2^*\big)+2(1+\gamma^2)|F_2|^2\right]$$

with scattering angle θ, $\beta=\sqrt{1-4m_\tau^2/s}$, $\gamma=\sqrt{s}/(2m_\tau)$

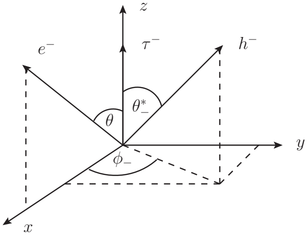
- Interference term $4\text{Re}\,\big(F_1F_2^*\big)$ sensitive to the sought two-loop effects
- Could be determined by fit to θ dependence
- But: need to measure total cross section at 10^{-6}

  $\hookrightarrow$ **can we use asymmetries instead**?
- Usual forward–backward asymmetry ($z=\cos\theta$)

$$\sigma_{\text{FB}} = 2\pi\left[\int_0^1 dz\frac{d\sigma}{d\Omega}-\int_{-1}^0 dz\frac{d\sigma}{d\Omega}\right]$$

alone does not help

Second attempt: normal asymmetry



- Idea: use **polarization information of the $\tau^\pm$**

  $\hookrightarrow$ semileptonic decays $\tau^\pm \to h^\pm \overset{(-)}{\nu}_\tau$, $h = \pi, \rho, \ldots$

  Bernabéu et al. 2007

- Polarization characterized by

$$\mathbf{n}^*_\pm = \mp\alpha_\pm \begin{pmatrix} \sin\theta^*_\pm \cos\phi_\pm \\ \sin\theta^*_\pm \sin\phi_\pm \\ \cos\theta^*_\pm \end{pmatrix} \qquad \alpha_\pm \equiv \frac{m_\tau^2 - 2m_{h^\pm}^2}{m_\tau^2 + 2m_{h^\pm}^2} = \begin{cases} 0.97 & h^\pm = \pi^\pm \\ 0.46 & h^\pm = \rho^\pm \end{cases}$$

  $\hookrightarrow$ angles in $\tau^\pm$ rest frame

- **Normal asymmetry**

$$A_N^\pm = \frac{\sigma_L^\pm - \sigma_R^\pm}{\sigma} \propto \operatorname{Im} F_2(s) \qquad \sigma_L^\pm = \int_\pi^{2\pi} d\phi_\pm \frac{d\sigma_{\text{FB}}}{d\phi_\pm} \quad \sigma_R^\pm = \int_0^\pi d\phi_\pm \frac{d\sigma_{\text{FB}}}{d\phi_\pm}$$

  $\hookrightarrow$ only get the imaginary part, **need electron polarization**

- **Transverse and longitudinal asymmetries** Bernabéu et al. 2007

$$A_T^{\pm} = \frac{\sigma_R^{\pm} - \sigma_L^{\pm}}{\sigma} \qquad A_L^{\pm} = \frac{\sigma_{\text{FB},\,R}^{\pm} - \sigma_{\text{FB},\,L}^{\pm}}{\sigma}$$

- Constructed based on helicity difference

$$d\sigma_{\text{pol}}^{S} = \frac{1}{2}\Big(d\sigma^{S\lambda}\big|_{\lambda=1} - d\sigma^{S\lambda}\big|_{\lambda=-1}\Big)$$

and then integrating over angles

$$\sigma_R^{\pm} = \int_{-\pi/2}^{\pi/2} d\phi_{\pm} \frac{d\sigma_{\text{pol}}^{S}}{d\phi_{\pm}} \qquad \sigma_L^{\pm} = \int_{\pi/2}^{3\pi/2} d\phi_{\pm} \frac{d\sigma_{\text{pol}}^{S}}{d\phi_{\pm}} \qquad \sigma_{\text{FB},\,R}^{\pm} = \int_0^1 dz_{\pm}^{*} \frac{d\sigma_{\text{FB, pol}}^{S}}{dz_{\pm}^{*}} \qquad \sigma_{\text{FB},\,L}^{\pm} = \int_{-1}^0 dz_{\pm}^{*} \frac{d\sigma_{\text{FB, pol}}^{S}}{dz_{\pm}^{*}}$$

- Linear combination

$$A_T^{\pm} - \frac{\pi}{2\gamma} A_L^{\pm} = \mp\alpha_{\pm} \frac{\pi^2\alpha^2\beta^3\gamma}{4s\sigma}\Big[\text{Re}\left(F_2 F_1^{*}\right) + |F_2|^2\Big]$$

isolates the interesting interference effect

How to make use of this?

Contributions to Re $F_2^{\text{eff}}(s)$	$s=0$	$s=(10\,\text{GeV})^2$
1-loop QED	1161.41	−265.90
e loop	10.92	−2.43
μ loop	1.95	−0.34
2-loop QED (mass independent)	−0.42	−0.24
HVP	3.33	−0.33
EW	0.47	0.47
total	1177.66	−268.77

$$\text{Re}\,F_2^{\text{eff}}((10\,\text{GeV})^2) \simeq \mp\frac{0.73}{\alpha_\pm}\Big(A_T^\pm - 0.56 A_L^\pm\Big)$$

- **Strategy**:
  - Measure effective $F_2(s)$

$$\text{Re}\,F_2^{\text{eff}} = \mp\frac{8(3-\beta^2)}{3\pi\gamma\beta^2\alpha_\pm}\Big(A_T^\pm - \frac{\pi}{2\gamma}A_L^\pm\Big)$$

  - Compare measurement to SM prediction for Re F_2^{eff}
  - Difference gives constraint on a_τ^{BSM}
  - A measurement of $A_T^\pm - \frac{\pi}{2\gamma}A_L^\pm$ at $\lesssim 1\%$ would already be competitive with current limits

How to make use of this?

- **Challenges**:
  - Cancellation in $A_T^\pm - \frac{\pi}{2\gamma} A_L^\pm$: $A_{T,L}^\pm = \mathcal{O}(1)$, difference $\mathcal{O}(\alpha)$
  - Two-loop calculation in SM see 2111.10378 for form factor and radiative corrections
  - Form factor only dominates for resonant $\tau^+\tau^-$ pairs

$$|H(M_\Upsilon)|^2 = \left(\frac{3}{\alpha}\text{Br}(\Upsilon \to e^+e^-)\right)^2 \simeq 100$$

  - However: continuum pairs dominate even at $\Upsilon(nS)$, $n = 1, 2, 3$, due to energy spread
  - Should consider $A_T^\pm$, $A_L^\pm$ also for nonresonant $\tau^+\tau^-$, but requires substantial investment in theory for SM prediction Gogniat, MH, Ulrich, work in progress