The Path Forward to N3LO

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SLAC NATIONAL ACCELERATOR LABORATORY

SLACmass retreat

SLAC, 12 May 2022

The Path Forward to N3LO

Based on the Snowmass White Paper:

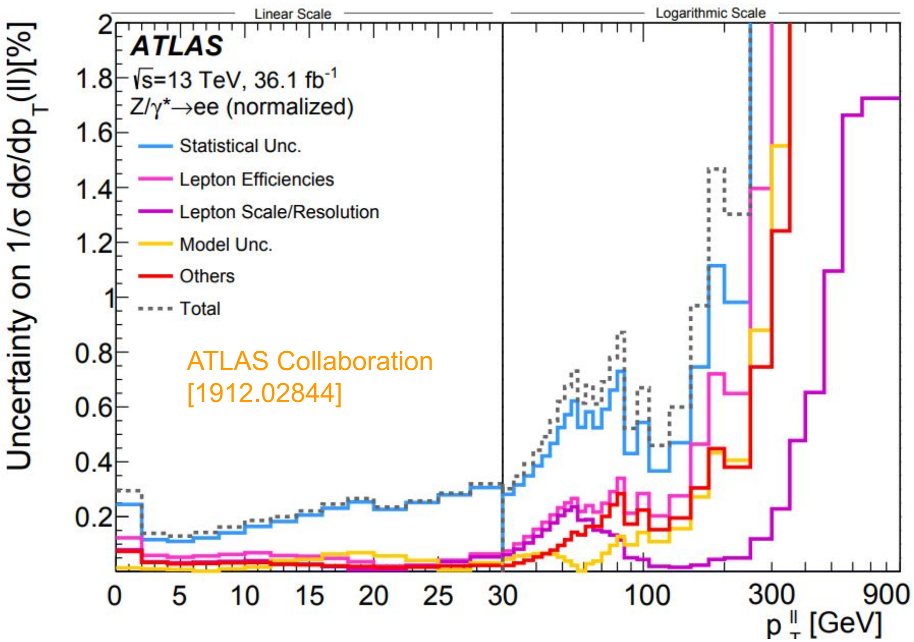
The Path forward to N^3LO

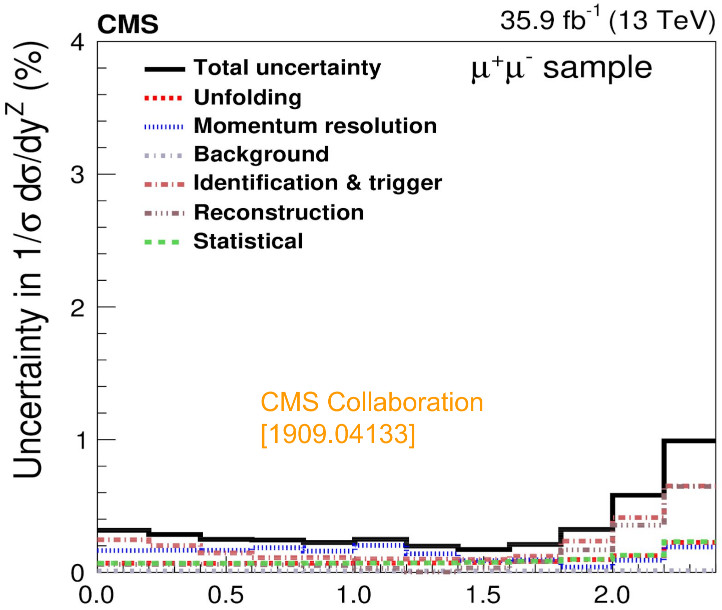
Fabrizio Caola, Wen Chen,
Claude Duhr, Xiaohui Liu,
Bernhard Mistlberger, Frank Petriello,
Gherardo Vita, Stefan Weinzierl

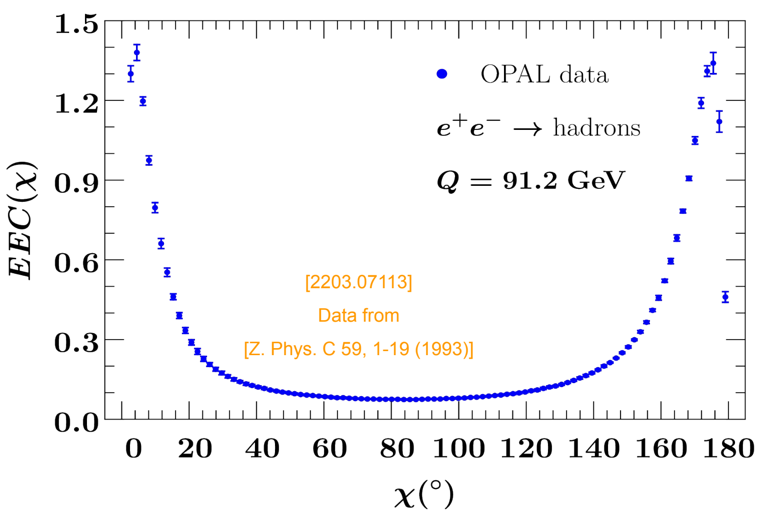
arXiv: 2203.06730

Testing the Standard Model at Colliders

- Experimental measurements of key benchmark processes have reached astonishing level of precision.



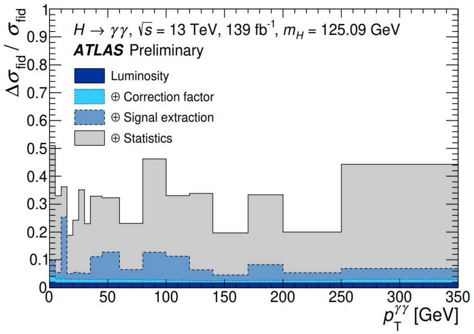


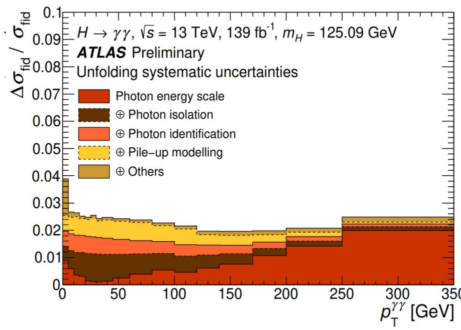


Ability to test the SM at (sub)-percent accuracy!

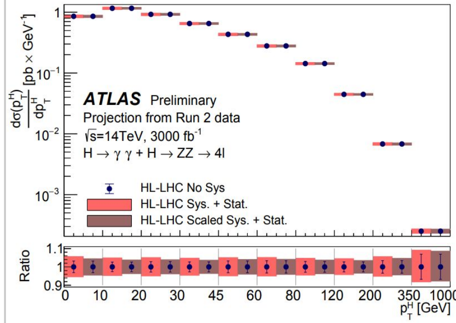
Testing the Standard Model at Colliders

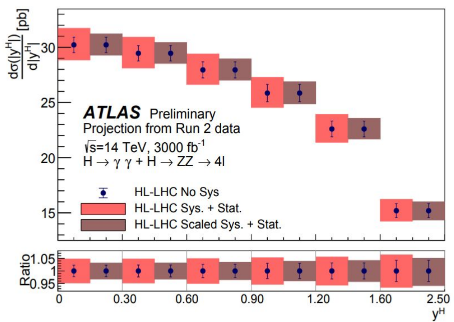
Higgs measurements at the moment are limited by statistics, but...

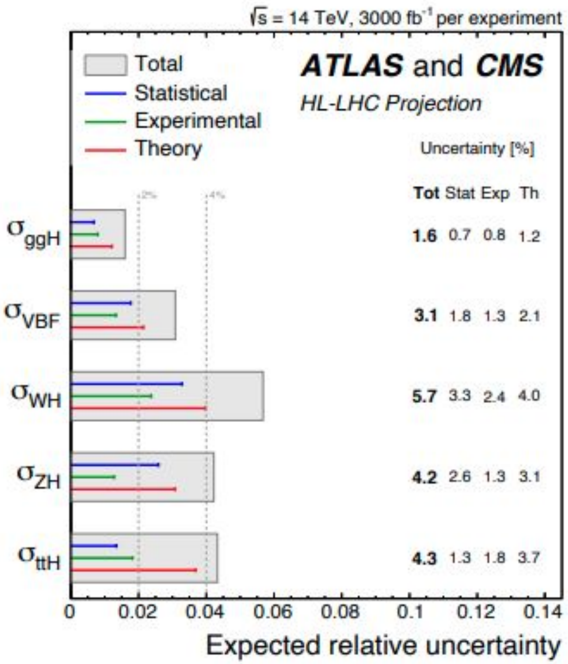




...statistical uncertainties will improve by a factor of 4-5 with High Luminosity LHC

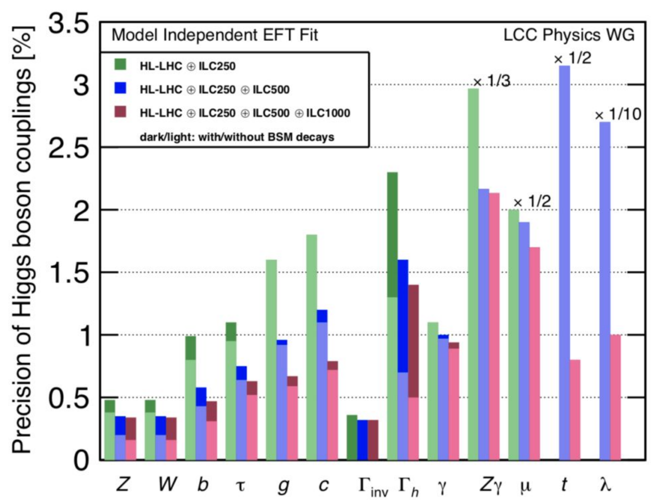


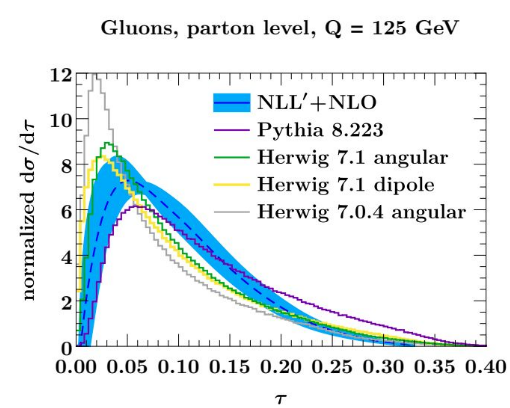




ATL-PHYS-PUB-2022-018

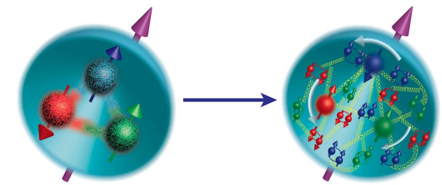
and things would get even more interesting with the ILC

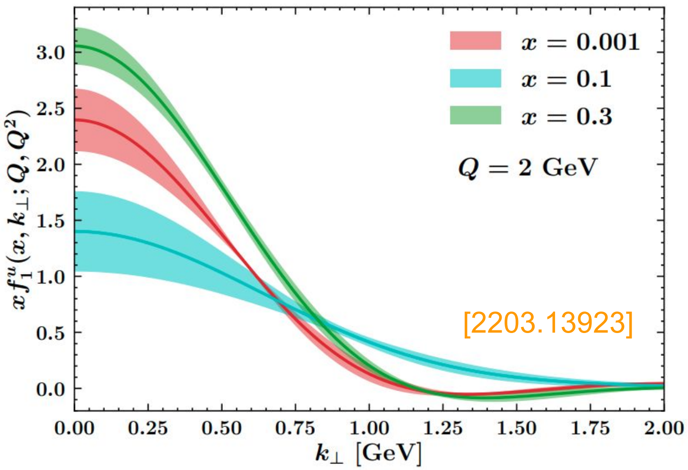


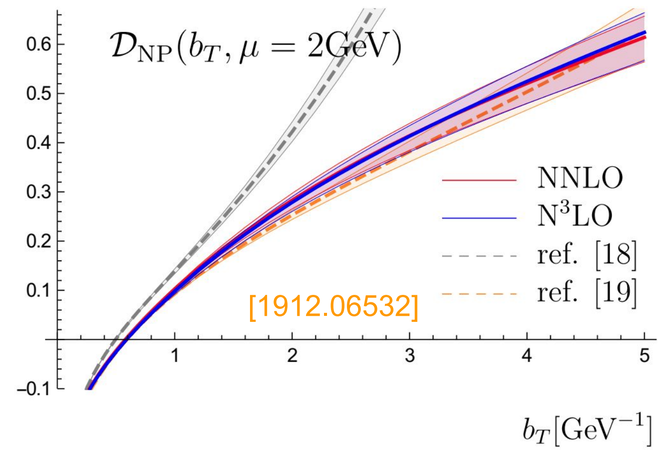


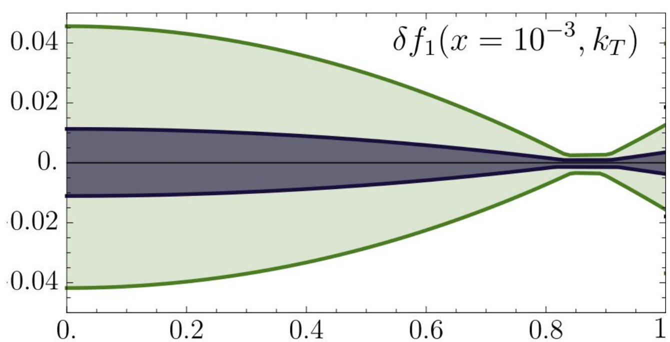
Testing the Standard Model at Colliders

Similarly, experimental measurements for TMD physics (3D tomography of the proton) will dramatically improve in the future thanks to the **Electron-Ion Collider**

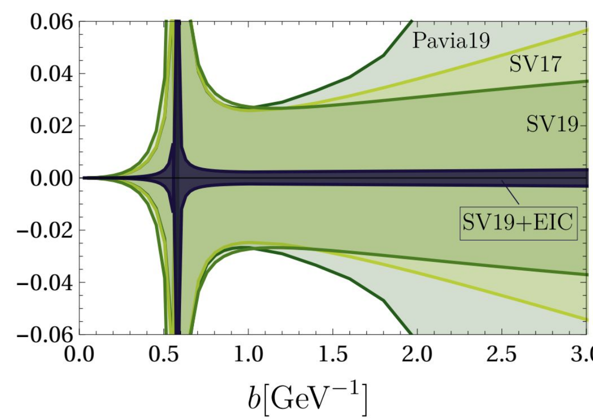








[EIC Yellow Report]



Improving Theoretical Predictions

To answer the fundamental questions we can probe at this level of accuracy, we should aim at comparable precision from the theory side!

Non Perturbative **Perturbative**

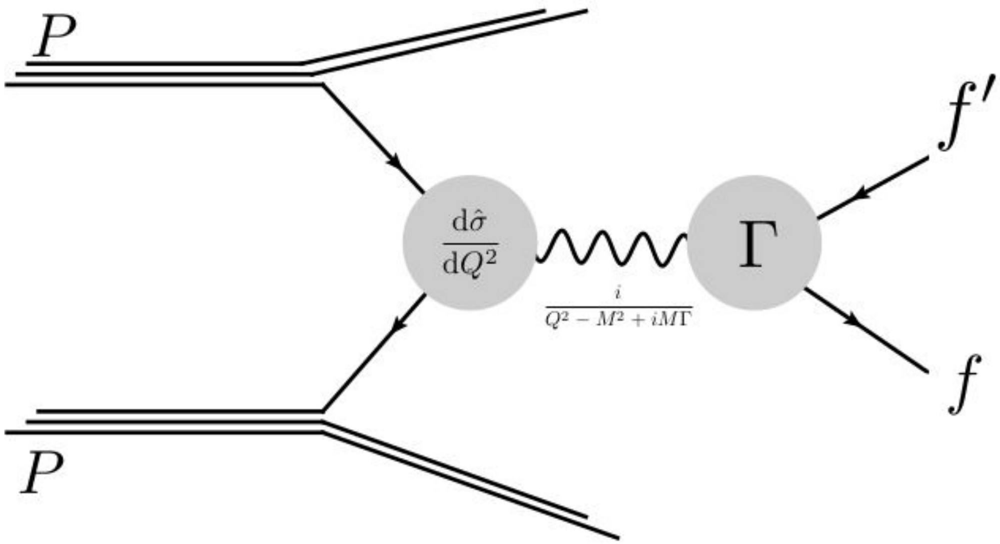
$$\sigma_{pp\to X} \sim \int f_a(x_1) f_b(x_2) \otimes \hat{\sigma}_{ab\to X}$$

$$\hat{\sigma}_{ab\to X} = \underbrace{\sigma_0}_{\text{LO}} + \underbrace{\alpha_s \sigma_1}_{\text{NLO}} + \underbrace{\alpha_s^2 \sigma_2}_{\text{NNLO}} + \underbrace{\alpha_s^3 \sigma_3}_{\text{N}^3\text{LO}} + \dots$$

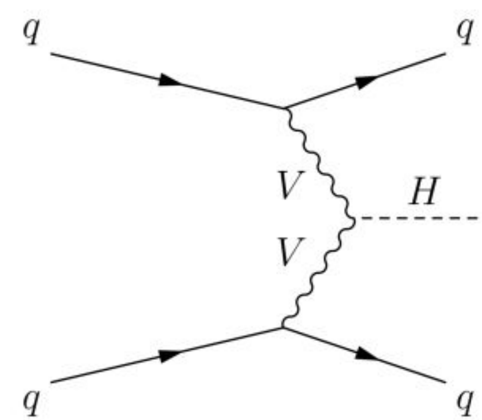
QCD perturbation theory

Status of N3LO Calculations

At the moment N3LO calculations obtained for very special case of **color singlet production**



$\gamma^* \; W \; Z \; H \quad bbH \quad H^* \; G \quad WH \; ZH$

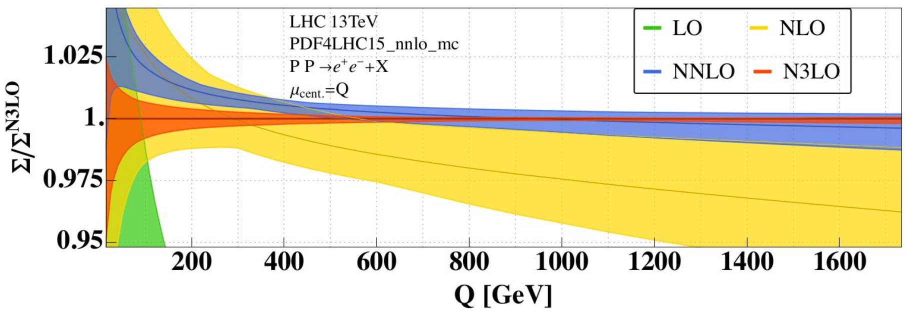


H, HH

- Mainly idealized observables such as inclusive cross section
- First results for differential/fiducial cross sections are coming out now

What have we learned so far

$$\hat{\sigma}_{ab\to X} = \underbrace{\sigma_0}_{\text{LO}} + \underbrace{\alpha_s\sigma_1}_{\text{NLO}} + \underbrace{\alpha_s^2\sigma_2}_{\text{NNLO}} + \underbrace{\alpha_s^3\sigma_3}_{\text{N}^3\text{LO}} + \ldots$$



CAVEAT!

Often times convergence turns out to be slower than naive estimate

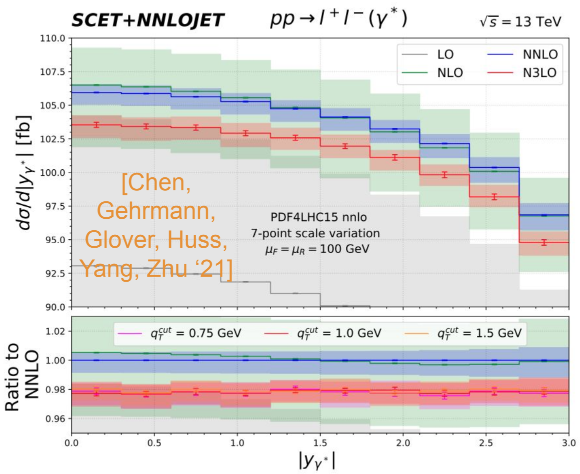
=> **N3LO gives few <u>percent</u> (not per-mille) shift**

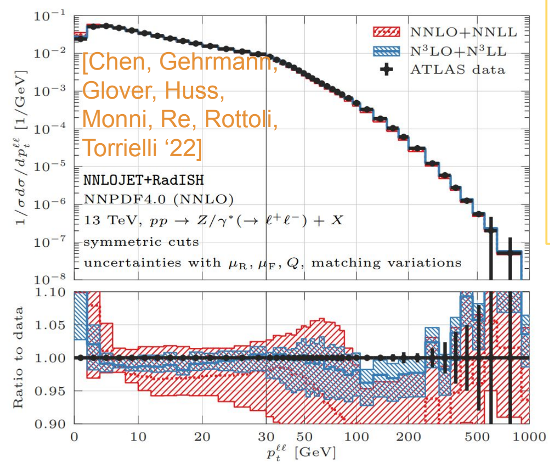
Example: N3LO/NNLO K-factors for inclusive Drell Yan and Higgs

	Q [GeV]	K-factor	δ(scale) [%]	$\delta(\text{PDF}+\alpha_S)$	δ(PDF-TH)
$gg \to$ Higgs	m_H	1.04	$^{+0.21\%}_{-2.37\%}$	$\pm3.2\%$	$\pm1.2\%$
$b\bar{b} \to$ Higgs	m_H	0.978	$^{+3.0\%}_{-4.8\%}$	$\pm8.4\%$	$\pm2.5\%$
NCDY	30	0.952	$^{+1.53\%}_{-2.54\%}$	$^{+3.7\%}_{-3.8\%}$	$\pm2.8\%$
	100	0.979	$^{+0.66\%}_{-0.79\%}$	$^{+1.8\%}_{-1.9\%}$	$\pm2.5\%$
CCDY(W^+)	30	0.953	$^{+2.5\%}_{-1.7\%}$	$\pm3.95\%$	$\pm3.2\%$
	150	0.985	$^{+0.5\%}_{-0.5\%}$	$\pm1.9\%$	$\pm2.1\%$
CCDY(W^-)	30	0.950	$^{+2.6\%}_{-1.6\%}$	$\pm3.7\%$	$\pm3.2\%$
	150	0.984	$^{+0.6\%}_{-0.5\%}$	$\pm2\%$	$\pm2.13\%$

What have we learned so far

$$\hat{\sigma}_{ab\to X} = \underbrace{\sigma_0}_{\text{LO}} + \underbrace{\alpha_s\sigma_1}_{\text{NLO}} + \underbrace{\alpha_s^2\sigma_2}_{\text{NNLO}} + \underbrace{\alpha_s^3\sigma_3}_{\text{N}^3\text{LO}} + \dots$$





CAVEAT!

Often times convergence turns out to be slower than naive estimate

=> **N3LO gives few <u>percent</u> (not per-mille) shift**

- Differential/normalized distributions follow similar pattern
- **Takeaway:**
  - N3LO gives few percent corrections
  - N3LO gets perturbative uncertainties to comparable size w.r.t. other uncertainties (PDFs, coupling constant, etc.)
  - **N3LO is required for percent level precision at the LHC**

How to go forward: Ingredients

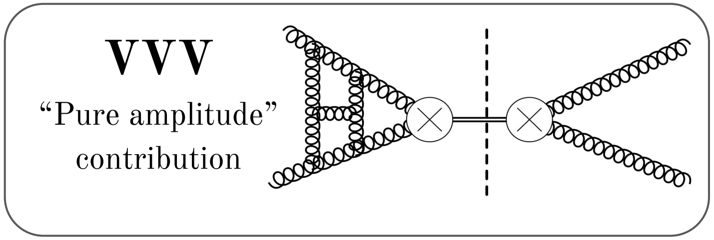
$$\sigma = f_1 \circ f_2 \circ \int d\Phi \, |M|^2$$

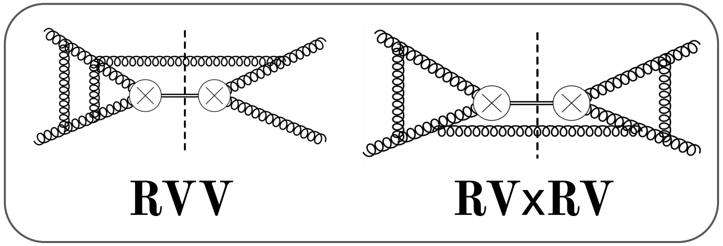
- Cross sections are obtained via **phase space integrals** over **amplitudes** (squared) convoluted with **Parton Distribution Functions (PDFs)**

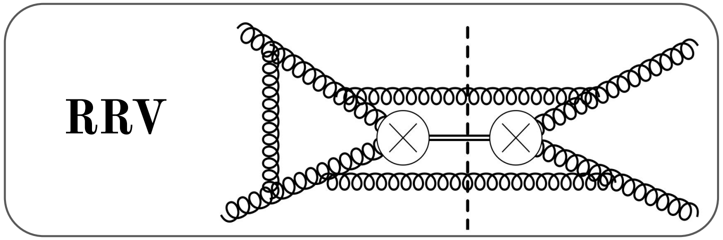
- Bottlenecks are present for each ingredient

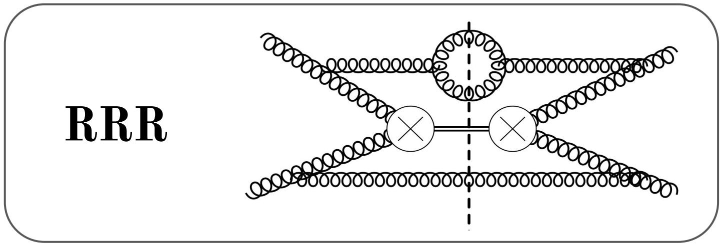
Example:

Higgs production at N3LO in gg









How to go forward: Amplitudes

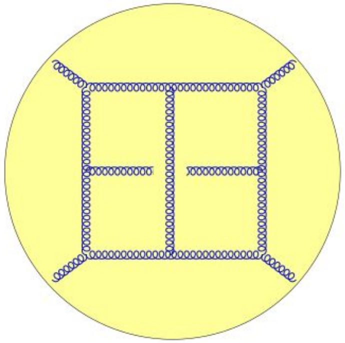
$$\sigma = f_1 \circ f_2 \circ \int d\Phi |M|^2$$

- For **amplitudes** we need
  - fast, stable numerical evaluation
  - compact expressions
  - Beyond Multiple PolyLogarithms - a new field of mathematical research!

$$\widetilde{\Gamma}\left(\begin{smallmatrix} n_1 & \cdots & n_r \\ c_1 & \cdots & c_r \end{smallmatrix}; z; \tau\right) = (2\pi i)^{n_1+\cdots+n_r-r} I_\gamma \left(\omega^{\text{Kronecker},z}_{n_1+1}(c_1,\tau), \ldots, \omega^{\text{Kronecker},z}_{n_r+1}(c_r,\tau); z\right)$$

- State of the art (amplitudes):

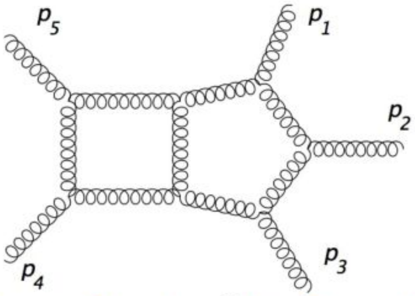
2 -> 2 at N3LO



Caola, Chakraborty, Gambuti, Mateuffel, Tancredi]

...

2 -> 3 at NNLO



[Abreu, Febres Cordero, Ita, Page, Slotnikov]

[Bayu, Badger, Brannum-Hansen, Peraro]

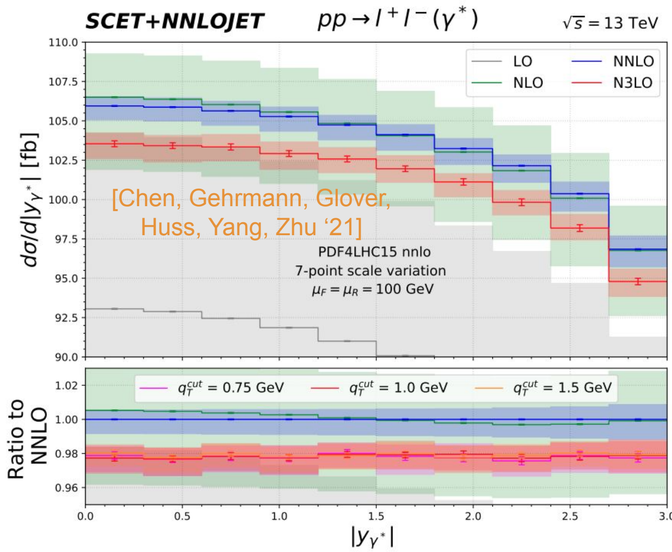
...

How to go forward: Phase Space

$$\sigma = f_1 \circ f_2 \circ \int d\Phi |M|^2$$

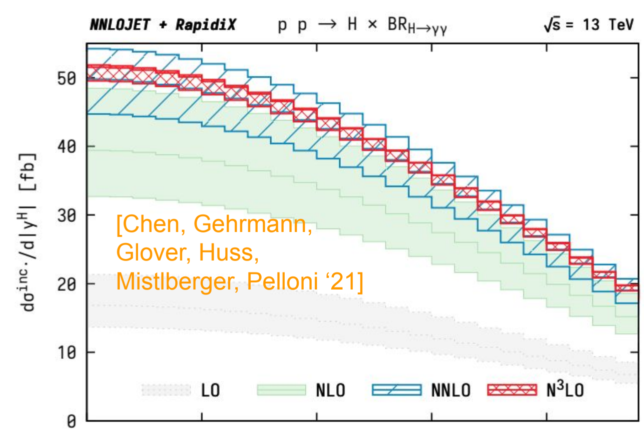
State of the art is **2 -> 1 at N3LO**

Slicing



and many more …

Subtraction



- For **phase space integrals:**
- Complexity of infrared singularities grows with loop order
- Numerically very expensive to handle: O(1 million) CPU hours for a computation
- Two Approaches:

Slicing

$$\sigma(X) = \underbrace{\int_0^{q_{T\,\mathrm{cut}}} dq_T \frac{d\sigma^{\mathrm{sing}}(X)}{dq_T}}_{\text{Below the cut region}} + \underbrace{\int_{q_{T\,\mathrm{cut}}} dq_T \frac{d\sigma(X)}{dq_T}}_{\text{Above the cut region}} + \underbrace{\Delta\sigma(X, q_{T\,\mathrm{cut}})}_{\text{Residual}}$$

Subtractions

$$\Delta\sigma^H_{\mathrm{NLO}} = \int [V\, d\Phi_H + \mathcal{S} d\Phi_{H+1}] + \int [R - \mathcal{S}]\, d\Phi_{H+1}.$$

How to go forward: Phase Space

$$\sigma = f_1 \circ f_2 \circ \int d\Phi |M|^2$$

Slicing

$$\sigma(X) = \int_0^{q_{T\,\mathrm{cut}}} \mathrm{d}q_T \frac{\mathrm{d}\sigma^{\mathrm{sing}}(X)}{\mathrm{d}q_T} + \int_{q_{T\,\mathrm{cut}}} \mathrm{d}q_T \frac{\mathrm{d}\sigma(X)}{\mathrm{d}q_T} + \Delta\sigma(X, q_{T\,\mathrm{cut}})$$

Below the cut region — **Above the cut region** — **Residual**

- Simpler than subtractions
- Numerically more challenging
- Below-the-cut-contribution via universal factorization theorem:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}Q^2 \mathrm{d}Y \mathrm{d}^2\vec{q}_T} = \sigma_0 \sum_{a,b} H_{ab}(Q^2, \mu) \int \frac{\mathrm{d}^2\vec{b}_T}{(2\pi)^2} e^{\mathrm{i}\,\vec{q}_T \cdot \vec{b}_T} \times \tilde{B}_a\left(x_1^B, b_T, \mu, \nu\right) \tilde{B}_b\left(x_2^B, b_T, \mu, \nu\right) S_q(b_T, \mu, \nu)$$

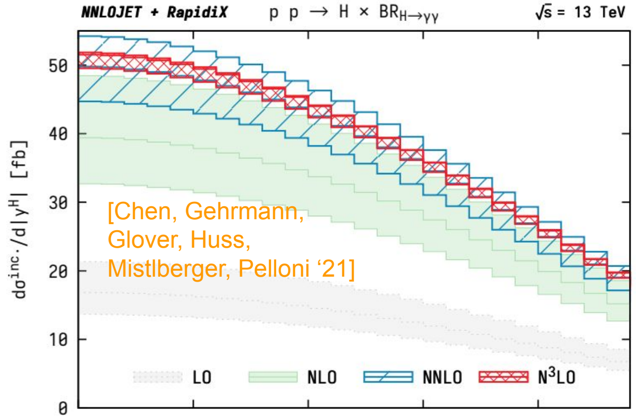
- All universal ingredients known for N3LO color singlet

[Ebert, Mistlberger, **GV**]
[Luo, Yang, Zhu, Zhu]

Subtractions

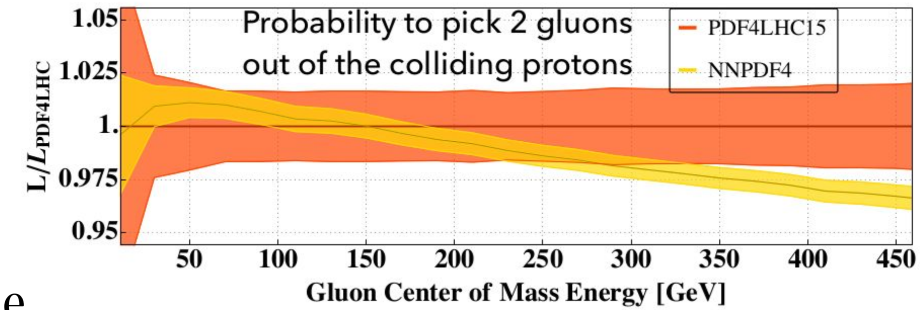
$$\Delta\sigma^H_{\mathrm{NLO}} = \int [V\ \mathrm{d}\Phi_H + \mathcal{S}\mathrm{d}\Phi_{H+1}] + \int [R - \mathcal{S}]\ \mathrm{d}\Phi_{H+1}$$

- Great success at NNLO
- Numerically efficient
- Complex to extend to N3LO



How to go forward: PDFs

$$\sigma = f_1 \circ f_2 \circ \int d\Phi |M|^2$$



Note: Parton (gluon) Luminosity already improving significantly thanks to LHC data

- Currently only NNLO PDFs available
- For **N3LO PDFs:**
  - Evolution of PDFs at N3LO: 4-loop splitting functions
    First results: [Moch,Ruijl,Ueda,Vermaseren,Vogt]
  - N3LO predictions for Global Dataset required.
  - Numerical capabilities to perform PDF fits.
  - EWK corrections / resummation / etc. in PDFs.

Dedicated Snowmass Whitepaper

Proton structure at the precision frontier

S. Alekhin, R. Ball, V. Bertone, C. Bissolotti, J. Blümlein, R. Boughezal, A. Buckley, F. G. Celiberto, A. Cooper-Sarkar, T. Cridge, C. Duhr, S. Forte, F. Giuli, A. Glazov, M. Guzzi, C. Gwenlan, L. Harland-Lang, T. J. Hobbs, S. Hoeche, J. Huston, H.-W. Lin, B. Mistlberger, S.-O. Moch, P. Nadolsky, E. Nocera, F. Olness, F. Petriello K. Rabbertz, C. Royon, J. Rojo, G. Schnell, K. Şimşek, M. Sutton, R. Thorne, M. Ubiali, G. Vita, J. H. Weber, K. Xie, C.-P. Yuan, B. Zhou,

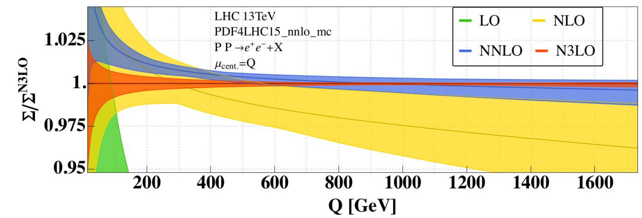
And many more things…

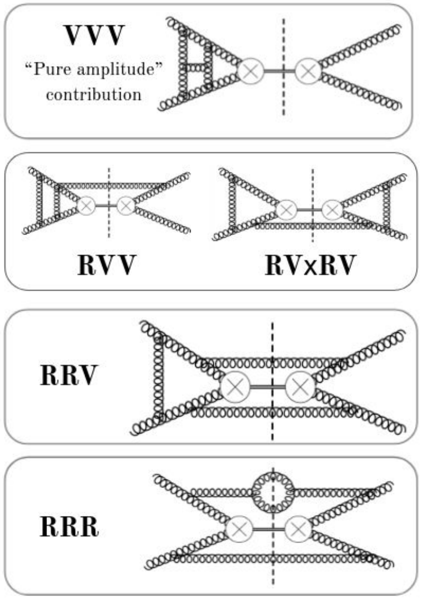
$$\sigma = f_1 \circ f_2 \circ \int d\Phi |M|^2 + \mathcal{O}(\Lambda^2/Q^2)$$

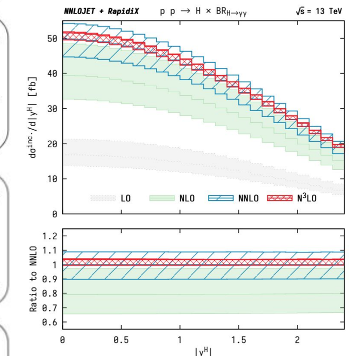
1. **Accessibility and User Friendliness:** Creating frameworks that make N^3LO (and NNLO) predictions easily accessible for comparison to experimental data.
2. **Corrections beyond QCD:** EWK and masses.
3. **Factorisation Violation at N^3LO:** tops, PDFs.
4. **Parton Showers:** Consistent combination of parton showers with fixed order perturbative computations at N^3LO.
5. **Resummation:** Complementing N^3LO computations and resummation techniques for infrared sensitive observables.
6. **Uncertainties:** Deriving / defining reliable uncertainty estimates for theoretical computations at the percent level.
7. **Beyond Leading Power Factorisation:** Exploring the limitations of leading power perturbative descriptions of hadron collision cross sections.

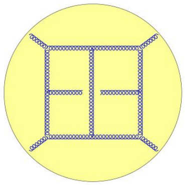
Conclusion

- ➢ The LHC will deliver a window into electroweak scale physics at the percent level.
- ➢ To fully exploit it we will need N3LO phenomenological predictions.
- ➢ Many advancements and community effort required over the next decade.
- ➢ Some major / immediate bottlenecks:
    - ■ Multiloop scattering amplitudes
    - ■ Phase space singularities
    - ■ N3LO PDFs









Caola, Chakraborty, Gambuti, Mateuffel, Tancredi]

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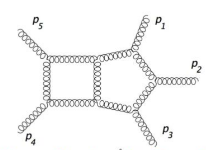

[Abreu, Febres Cordero, Ita, Page, Slotnikov]

[Bayu, Badger, Brannum-Hansen, Peraro]

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$$\sigma(X) = \int_0^{q_{T\,\mathrm{cut}}} \mathrm{d}q_T \frac{\mathrm{d}\sigma^{\mathrm{sing}}(X)}{\mathrm{d}q_T} + \int_{q_{T\,\mathrm{cut}}} \mathrm{d}q_T \frac{\mathrm{d}\sigma(X)}{\mathrm{d}q_T} + \Delta\sigma(X, q_{T\,\mathrm{cut}})$$

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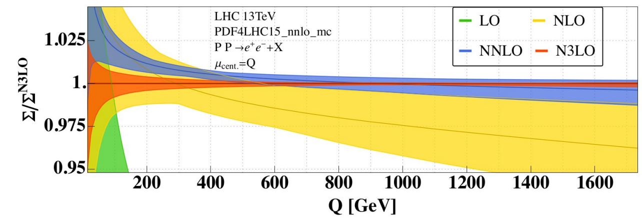

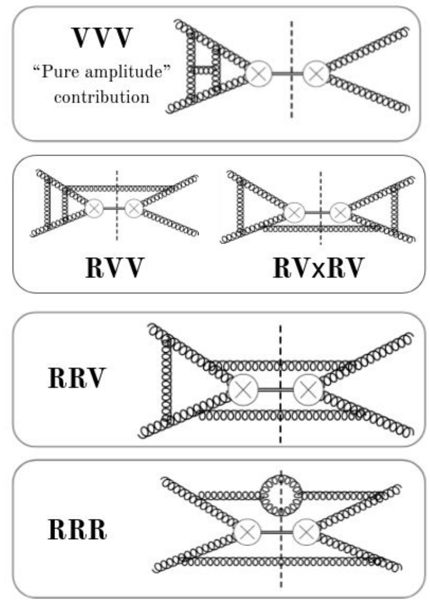

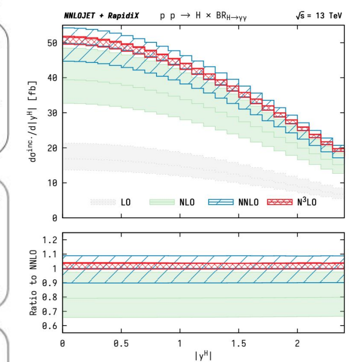

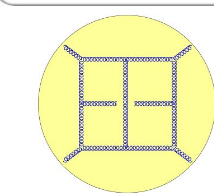

Caola, Chakraborty, Gambuti, Mateuffel, Tancredi]
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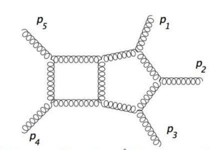


[Abreu, Febres Cordero, Ita, Page, Slotnikov]
[Bayu, Badger, Brannum-Hansen, Peraro]
...

THANK YOU!

$$\sigma(X) = \int_0^{q_{T\,\mathrm{cut}}} \mathrm{d}q_T \frac{\mathrm{d}\sigma^{\mathrm{sing}}(X)}{\mathrm{d}q_T} + \int_{q_{T\,\mathrm{cut}}} \mathrm{d}q_T \frac{\mathrm{d}\sigma(X)}{\mathrm{d}q_T} + \Delta\sigma(X, q_{T\,\mathrm{cut}})$$

Backup

Differential Distributions via Slicing

- Cross sections have IR divergences due to soft and collinear radiation at intermediate steps of the calculation.
- This complicates automatizing higher order calculations
- One way of dealing with this problem semi-numerically is to use **slicing methods**

q_T subtraction

[Catani, Grazzini '07]

N-Jettiness subtraction

[Boughezal, Focke, Liu, Petriello '15]
[Gaunt, Stahlhofen, Tackmann, Walsh '15]

- Find an observable that isolates the Born configuration of a given process to the region where the observable vanishes.

$$\sigma(X) = \int \mathrm{d}q_T \frac{\mathrm{d}\sigma(X)}{\mathrm{d}q_T} = \int_0^{q_{T\,\mathrm{cut}}} \mathrm{d}q_T \frac{\mathrm{d}\sigma(X)}{\mathrm{d}q_T} + \int_{q_{T\,\mathrm{cut}}} \mathrm{d}q_T \frac{\mathrm{d}\sigma(X)}{\mathrm{d}q_T} \,.$$

$$\frac{\mathrm{d}\sigma(X)}{\mathrm{d}q_T} = \underbrace{\frac{\mathrm{d}\sigma^{\mathrm{sing}}(X)}{\mathrm{d}q_T}}_{\sim 1/q_T} + \underbrace{\sum_{i>0} \frac{\mathrm{d}\sigma^{(i)}(X)}{\mathrm{d}q_T}}_{\text{integrable as } q_T \to 0}$$

Differential Distributions via Slicing

- Cross sections have IR divergences due to soft/collinear modes at intermediate steps
- One way of dealing with this problem semi-numerically is to use **slicing methods**

q_T subtraction

[Catani, Grazzini '07]

N-Jettiness subtraction

[Boughezal, Focke, Liu, Petriello '15]
[Gaunt, Stahlhofen, Tackmann, Walsh '15]

- Find observable that isolates Born configuration to region where observable vanishes

- Organize cross section as:

(example using q_T subtraction)

$$\sigma(X) = \int_0^{q_{T\,\mathrm{cut}}} \mathrm{d}q_T \frac{\mathrm{d}\sigma^{\mathrm{sing}}(X)}{\mathrm{d}q_T} + \int_{q_{T\,\mathrm{cut}}} \mathrm{d}q_T \frac{\mathrm{d}\sigma(X)}{\mathrm{d}q_T} + \Delta\sigma(X, q_{T\,\mathrm{cut}})$$

Below the cut region:

- Singular distribution
- Contains most complicated cancellation of IR divergences
- Control it analytically via factorization theorems

Above the cut region:

- Resolved extra radiation
- No events in Born configuration
- Lower number of loops
- Calculate numerically and/or with lower order subtraction schemes

Residual error:

Non singular terms from below the cut.
Can be systematically reduced by analytically computing subleading power corrections

Differential Distributions via Slicing

- Extremely successful program for many color singlet LHC processes at NNLO

$pp \to Z$, $pp \to W$, $pp \to H$, $pp \to \gamma\gamma$, $pp \to Z\gamma$, $pp \to W\gamma$, $pp \to ZZ$, $pp \to WW$, $pp \to WZ$

[Matrix collaboration]

- With N-Jettiness ability to tackle also processes with jets in the final state

[Boughezal, Focke, Liu, Petriello + Campbell, Ellis, Giele '15, '16]

[Campbell, Ellis, Williams '16]

[Mondini, Williams '21]

[Campbell, Ellis, Seth '19]

- Error due to higher order terms in q_T expansion

$$\Delta\sigma(X, q_{T\,\mathrm{cut}}) \equiv \sum_{i>0} \int_0^{q_{T\,\mathrm{cut}}} \mathrm{d}\tau \frac{\mathrm{d}\sigma^i(X)}{\mathrm{d}q_T}$$

- In principle reduced by pushing cut to small values, in practice: tradeoff between numerical stability of above the cut result and size of power corrections
- Interesting prospects of improving them by analytically including power corrections

Singular Region for q_T Slicing

- **Singular region** (i.e. below the cut) can be understood at all orders as

Leading power **factorization** for Transverse-Momentum Distributions in pp

$$\frac{d\sigma}{dQ^2 dY\, d^2\vec{q}_T} = \sigma_0 \sum_{i,j} H_{ij}(Q^2, \mu) \int d^2\vec{b}_T\, e^{i\vec{q}_T\cdot\vec{b}_T}\, \tilde{B}_i\Big(x_1^B, b_T, \mu, \frac{\nu}{\omega_a}\Big) \tilde{B}_j\Big(x_2^B, b_T, \mu, \frac{\nu}{\omega_b}\Big) \tilde{S}(b_T, \mu, \nu)$$

q_T Beam Functions

- At each order:
  - Log-enhanced terms (predicted by RGE/anomalous dims. and lower order results)
  - Boundary values (non-log enhanced terms, need explicit calculation)

- Boundary value for Hard and Soft are **constants**.
  - Known at N3LO for Hard since 2010 and for Soft since 2016. [Li, Zhu]

[Gehrmann, Glover, Huber, Ikizlerli, Studerus]

- **Beam function** boundary values are **full functions** (of the collinear splitting variable)
  - More complicated objects.
  - Different for quark vs gluons

Last missing ingredients for qT subtraction at N3LO

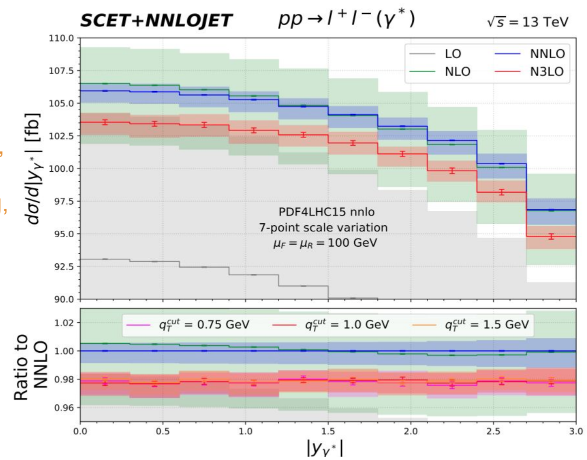
[Ebert, Mistlberger, Vita]

[Luo, Yang, Zhu, Zhu]

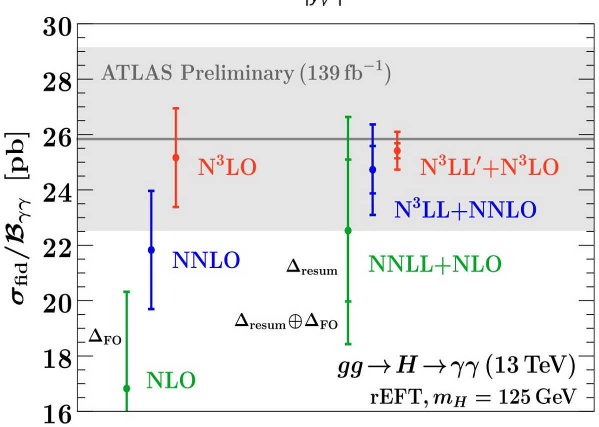
Slicing at N3LO

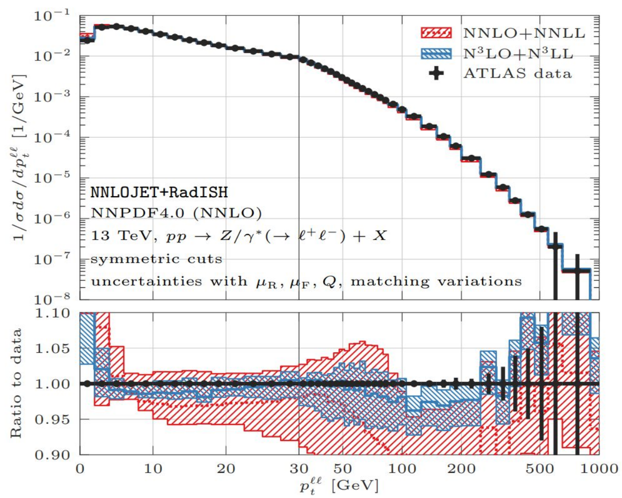
- **q_T beam functions** at N3LO were **last missing ingredient** for:
  - q_T subtraction for differential and fiducial Drell-Yan and Higgs production at N3LO
  - q_T resummation at N3LL`
- Many new exciting phenomenological results at N3LO employing them!

[Chen, Gehrmann, Glover, Huss, Yang, Zhu '21]



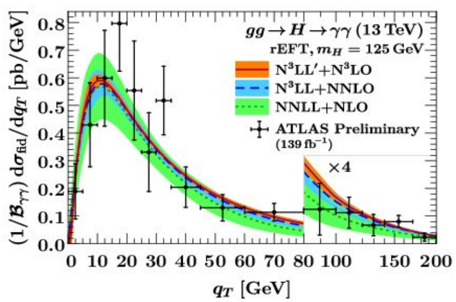
[Billis, Dehnadi, Ebert, Michel, Tackmann '21]

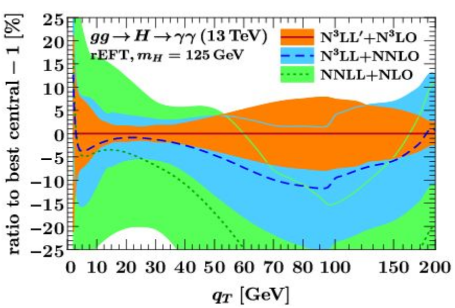




[Chen, Gehrmann, Glover, Huss, Monni, Re, Rottoli, Torrielli '22]

[Billis, Dehnadi, Ebert, Michel, Tackmann '21]





And many more:

[Camarda, Cieri, Ferrera '21]

[Neumann '21]

[Re, Rottoli, Torrielli '21]

[Ju, Schönherr '21]