Modeling Vector and Axial Nucleon Form Factors with Bayesian Neural Networks

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Problem

Axial and Vector Form Factors (FF) of the nucleon:

- quasielastic charged current and elastic neutral current νN and ν-nucleus scattering cross-sections parameterized by vector and axial nucleon form factors.
- information about electroweak structure of the nucleon hidden in FF
- empirical information from:
  - elastic electron scattering data
  - quesielastic ν scattering data
- nucleon form factors: real-valued functions (scalars) depending on Q^2

Task

- obtain the G_{Ep} (electric), G_{Mp} magnetic protons FF
- obtain F_A axial FF

Neural Networks and Bayesian Statistics

Problems

- the choice of FF parametrization affects the results of the analysis: prediction of the uncertainties etc.
- bias-variance trade-off: **too simple models under-fit the data**, **too complex models tend to over-fit the data**

Method: Bayesian Neural Networks (BNN):

- Bayesian approach naturally embodies Occam's razor: **penalizes too complex models and favor simpler approaches** hence good generalization abilities
- Form Factors parametrized by Feed Forward Neural Networks in Multi Layer Perceptron (MLP) configuration
- Bayesian framework for Neural Networks: following MacKay's

$\rightarrow$ Estimate of the uncertainties of the model predictions

$\rightarrow$ validation data set is not required

$\rightarrow$ Quantitative comparison of different models

Bayesian model

i) Consider data $\mathcal{D}$ and Neural Network, $\mathcal{N}(\{w_i\})$

ii) two conditional probabilities: prior and likelihood

iii) from Bayes theorem **Posterior**

$$\underbrace{\mathbf{P}(\{\mathbf{w_i}\}|\mathcal{D},\mathcal{N})}_{posterior} = \frac{\overbrace{P(\mathcal{D}|\{w_i\},\mathcal{N})}^{likelihood}\,\overbrace{P(\{w_i\}|\mathcal{N})}^{prior}}{\underbrace{P(\mathcal{D}|\mathcal{N})}_{evidence}} \quad (1)$$

- **Evidence** for model $\mathcal{N}$

$$P(\mathcal{N}|\mathcal{D}) = \frac{P(\mathcal{D}|\mathcal{N})P(\mathcal{N})}{P(\mathcal{D})} \sim P(\mathcal{D}|\mathcal{N})P(\mathcal{N}) \sim P(\mathcal{D}|\mathcal{N}) \quad (2)$$

so evidence ranks models

- likelihood

$$-2\ln\mathcal{P}(\mathcal{D}|\{w_i\}),\mathcal{N})\sim\chi^2_{ex}(\mathcal{D},\mathcal{N},\{w_i\})\qquad(3)$$

- prior

$$\mathcal{P}(\{w_i\}|\alpha,\mathcal{N})\quad\sim\quad\exp\left(-\frac{\alpha}{2}E_w\right),\quad E_w=\sum_{k=1}^{W}w_k^2\qquad(4)$$

  α regularizer (hyperparameter)

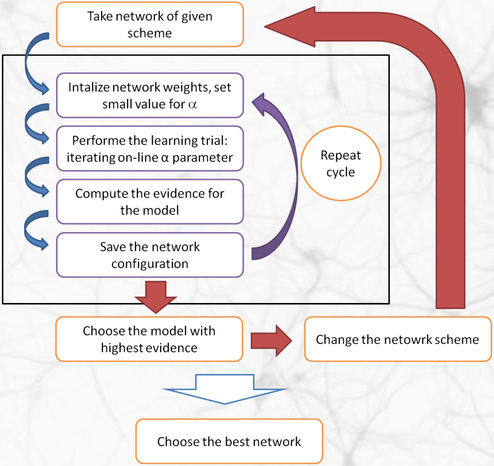
- posterior

$$-2\ln P(\{w_i\}|\mathcal{D},\mathcal{N},\alpha)\sim(\chi^2_{ex}(\mathcal{D},\{w_i\})+\alpha E_w)=\mathcal{E}(\mathcal{D},\{w_i\}),\qquad(5)$$

  $\mathcal{E}(\mathcal{D},\{w_j\})$ has a minimum at $\{w_j\}_{MP}$ and α_{MP}

- α_{MP} established during the training
- **1 step of inference:** the posterior is maximized
- **2 step of inference:** the evidence for each model is calculated to choose the best model!

Computation Scheme



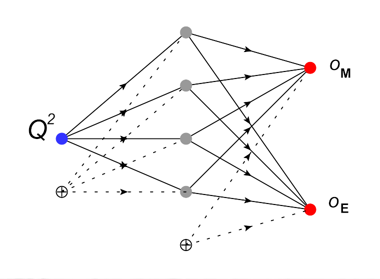
- every iteration step α's are changed (Hessian matrix is calculated and its eigenvalues)

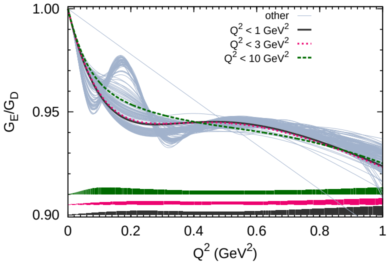
- C++ library developed by K.M. Graczyk and C. Juszczak
- user defined error function
- symbolic derivatives (but also numerical derivatives available)
- several optimization algorithms: gradient descent, QuickProp, RPROP (in several configurations, Adam, **Levenberg-Marquardt** (with approximate and exact Hessian)

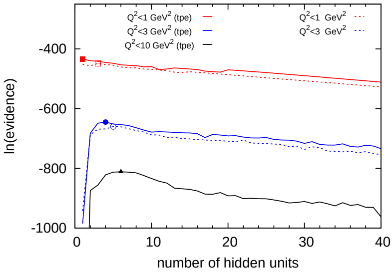
Electric and Magnetic Form Factors of the Proton: Graczyk and Juszczak, PRC90, 054334 (2014)

- 27 elastic ep scattering cross section data sets (27 normalization parameters in the fit)
- 15 polarization transfer data sets (ratios $\mu_p G_{Ep}/G_{Mp}$)

$$G_{Ep} = (1 - Q^2 o_E) G_D, \quad (6)$$
$$G_{Mp} = \mu_p (1 - Q^2 o_M) G_D, \quad (7)$$







Axial Form Factor

Alvarez-Ruso, Graczyk and Saul-Sala, PRC99 (2019), 025204

- **ANL** Bubble Chamber Data: $\nu_\mu + d \to \mu^- + p + p$ (PRD26, 537): distribution of events: $Q^2 \in (0.05, 2.5)$ GeV^2
- The least square-function:

$$\chi^2_{ex} = \chi^2_{\rm ANL} + \chi^2_{g_A} \tag{8}$$

where

$$\chi^2_{\rm ANL} = \sum_{i=k}^{n_{\rm ANL}} \frac{\left(N_i - pN_i^{th}\right)^2}{N_i} + \left(\frac{1-p}{\Delta p}\right)^2 \tag{9}$$

Δp – systematic uncertainty for $\# N$ of events

$$N_i^{th} = \int_0^\infty dE_\nu \frac{\frac{d\sigma}{dQ^2}(E_\nu, F_A)}{\sigma(E_\nu, F_A)} \frac{dN}{dE_\nu}, \quad \sigma(E_\nu, F_A) = \int_{min}^{max} \frac{d\sigma}{dQ^2}(E_\nu, F_A) dQ^2 \tag{10}$$

- $F_A(Q^2 = 0)$,

$$\chi^2_{g_A} = \left(\frac{F_A(0) - g_A}{\Delta g_A}\right)^2 \tag{11}$$

g_A and Δg_A from PDG

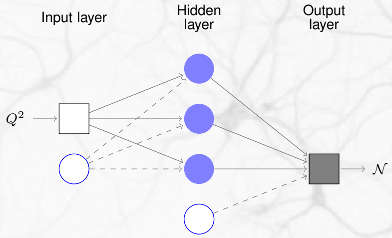
- Cross section modified by nuclear corrections for deuterium.

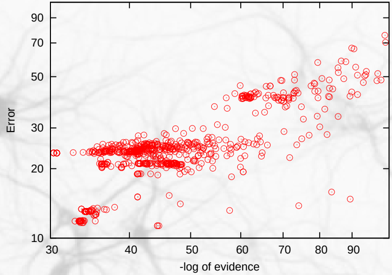
Analysis

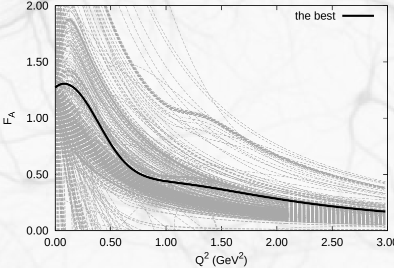
(i) BIN0: all ANL bins included

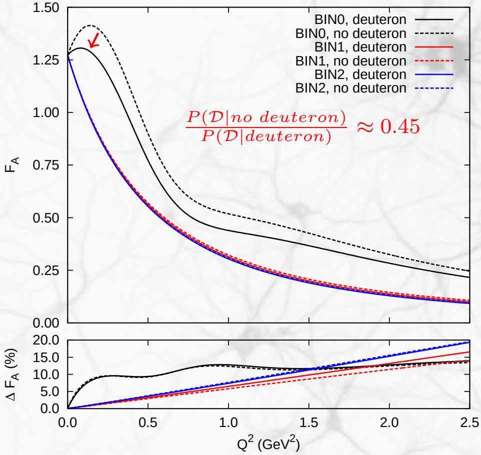
(ii) BINk: where $k = 1$ or $k = 2$: ANL bins without the first k bins

$$F_A(Q^2) = F_A^{\text{dipole}}(Q^2) \times \mathcal{N}_M(Q^2; \{w_i\}) \quad (12)$$



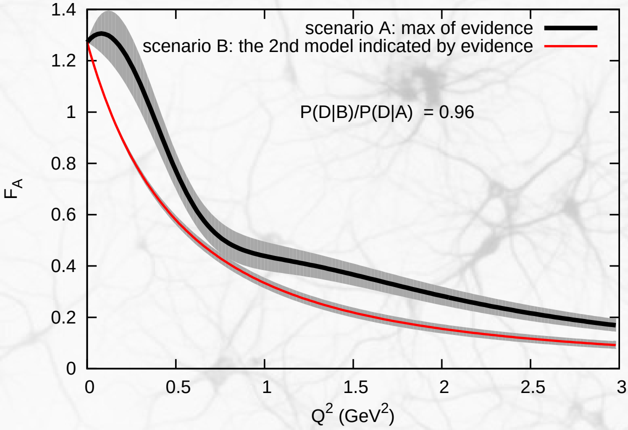






- For BIN0 The slope of F_A at $Q^2 = 0$ not consistent with other determinations...
- fits of BIN1 and BIN2 data consistent with original ANL analysis
- deuteron correction important for the first bin
- Possible explanation:
  - a low quality of the measurements at low-Q^2 due to low and not well understood efficiency
  - an improper description of the nuclear corrections
  - the actual value of the slope $dF_A(Q^2 = 0)/dQ^2$ might not be properly estimated because of the lack of very low-Q^2 data

two competitive scenarios: on the edge of dipole...



- small effects are important
- there is a tension between the first bin and constraint for F_A at $Q^2 = 0$
- assumption: $dF_A(Q^2 = 0)/dQ^2 < 0 \rightarrow$ the fit in agreement with dipole shape...

Final Remarks

- First Bayesian analyses of the electron-proton and neutrino-deuteron scattering data presented.

The method allows to:

- analyze small data sets
- reduce the model dependence of final results
- compare quantitatively different models

* Calculations done in WrocÂław Centre for Networking and Supercomputing, Grant No. 268 (http://www.wcss.wroc.pl)