Exercises for neutrino mass and testing seesaw

Tao Han, SSI 2019, SLAC (Aug. 21, 2019)

Problem 1: Neutrino mass in the simplest extension of the SM

The leptonic content in the simplest extension of the SM includes three generations of left-handed $SU(2)_L$ doublets and charged right-handed singlets, plus n new right-handed neutral singlets

$$L_{aL} = \begin{pmatrix} \nu_a \\ l_a \end{pmatrix}_L, \quad l_{aR}, \quad \text{and } N_{bR}, \tag{1}$$

where $a = 1, 2, 3$ and $b = 1, 2, \cdots, n$ ($n = 3$ for three massive neutrinos). Constructing the gauge-invariant Yukawa interactions with the Higgs field

$$-\mathcal{L}_Y = \left(\sum_{a,b=1}^{3} f^l_{ab}\ \overline{L_{aL}}\ H l_{bR} + \sum_{a=1}^{3}\sum_{b=1}^{n}\ f^\nu_{ab}\ \overline{L_{aL}}\ \hat{H} N_{bR} \right) + h.c. \tag{2}$$

where H is the SM Higgs doublet

$$H = \begin{pmatrix} H^+ \\ H^0 \end{pmatrix} = \begin{pmatrix} \phi^+ \\ (h + v + i\phi^0)/\sqrt{2} \end{pmatrix} \tag{3}$$

and $\hat{H} = i\tau_2 H^*$ (τ_2 is the second Pauli matrix). After the Higgs field develops a vacuum expectation value $\langle H^0 \rangle = v/\sqrt{2}$, neutrinos obtain Dirac masses.

(1). **Express the neutrino Dirac masses and derive the Feynman rule for the Higgs-neutrino coupling vertex $h\bar{\nu}\nu$.**

It is also possible for the singlet neutrinos to have a Majorana mass term. The full neutrino mass terms as well as the diagonalized eigenvalues can be expressed as

$$\begin{aligned} -\mathcal{L}^\nu_m &= \frac{1}{2}\left(\sum_{a=1}^{3}\sum_{b=1}^{n} (\overline{\nu_{aL}}\ m^\nu_{ab}\ N_{bR} + \overline{N^c_{bL}}\ m^{\nu *}_{ba}\ \nu^c_{aR}) + \sum_{b,b'=1}^{n}\ \overline{N^c_{bL}}\ B_{bb'}\ N_{b'R} \right) + \text{h.c.} \\ &= \frac{1}{2}\left(\sum_{m=1}^{3} m_{\nu_m}\ \overline{\nu_{mL}}\ \nu^c_{mR} + \sum_{m'=4}^{3+n} M_{N_{m'}}\ \overline{N^c_{m'L}}\ N_{m'R} \right) + \text{h.c.} \end{aligned} \tag{4}$$

with the mixing relations between the gauge and mass eigenstates

$$\nu_{aL} = \sum_{m=1}^{3} U_{am}\nu_{mL} + \sum_{m'=4}^{3+n} V_{am'} N^c_{m'L}, \tag{5}$$

$$UU^\dagger + VV^\dagger = I. \tag{6}$$

Parametrically, $U^\dagger U \sim \mathcal{O}(1)$ and $V^\dagger V \sim \mathcal{O}(m_\nu/M_N)$. The charge-conjugate state is defined as $\psi^c = C\bar{\psi}^T$ ($\overline{\psi^c} = \psi^T C$).

(2). Starting from the universal charged-current gauge interactions

$$-\mathcal{L} = \left(\frac{g}{\sqrt{2}} W_\mu^+ \sum_{a=1}^{3} \overline{\nu_{aL}}\ \gamma^\mu l_{aL} + h.c. \right), \tag{7}$$

derive the gauge interaction vertex between the charged leptons and the heavy neutrinos $W^\pm \bar{N} \ell^\mp$.

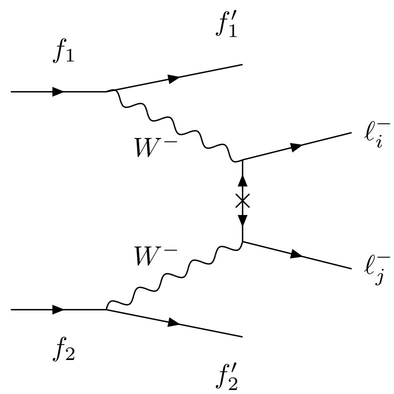

FIG. 1: A generic diagram for $\Delta L = 2$ processes via Majorana neutrino exchange.

Problem 2: Testing the Majorana nature via $\Delta L = 2$ processes

If there exists a massive Majorana neutrino labelled by N_4, it will mediate $\Delta L = 2$ processes as depicted in Fig. 1. The Feynman amplitude of the leptonic part can be written as

$$\mathcal{M}_{lep}^{\mu\nu} = \frac{g^2}{2} V_{\ell_1 4} V_{\ell_2 4}\ m_4 \frac{\overline{u(\ell_i)}\gamma^\mu \gamma^\nu P_R v(\ell_j)}{q^2 - m_4^2 + i\Gamma_{N_4} m_4} + (i \to j). \tag{8}$$

The full amplitude for a $\Delta L = 2$ process as in Fig. 1 can be casted into the leptonic part and the fermionic currents via the W-exchange

$$i\mathcal{M} = (\mathcal{M}_{lep})^{\mu\nu} J_\mu(f_1' f_1) J_\nu(f_2' f_2). \tag{9}$$

One sees that by coupling fermion currents $(f_1' f_1)$ and $(f_2' f_2)$ to the W bosons as in Fig. 1, and arranging the initial and final states properly, one finds various physical processes that can be experimentally searched for.

(1). The best known example is the neutrinoless double-beta decay ($0\nu\beta\beta$), $nn \to p^+ p^+ e^- e^-$, which proceeds via the parton-level subprocess $dd \to uu\ W^{-*} W^{-*} \to uu\ e^- e^-$ as seen in Fig. 1. Consider the characteristic amplitude factor

$$\mathcal{M} \sim f(q) \frac{(V \text{ or } U)_{e4} (V \text{ or } U)_{e4}\ m_4}{q^2 - m_4^2},$$

where $f(q)$ is an energy-dependent nuclear form factor with a typical momentum transfer $q \sim 1$ MeV. **In the two limiting cases, $m_4^2 \ll q^2$ and $m_4^2 \gg q^2$, discuss what physical parameters the $0\nu\beta\beta$ results will probe.**

(2). An important observation is that when the heavy neutrino mass is kinematically accessible, a process may undergo resonant production of the heavy neutrino in an intermediate

s-channel. The transition rate can be substantially enhanced and goes like

$$\frac{\Gamma(N_{m'} \to i)\ \Gamma(N_{m'} \to f)}{m_{N_{m'}} \Gamma_{N_{m'}}}, \tag{10}$$

where i, f refer to the initial and final state during the transition. One of the interesting processes is the tau decay such as $\tau^- \to \ell^+ M_1^- M_2^-$ where the light mesons M_1, M_2 are π, K. **Try to construct the decay amplitude according to Eqs. (8) and (9) in terms of the meson decay constants, flavor-mixing matrix elements for the mesons and leptons.** (Do not need to evaluate the matrix element and the decay rate.)

Along this line, list as many other possible $\Delta L = 2$ processes as you possibly can think of to test the Majorana nature of the neutrinos.

Reference to the problem solutions:

I. NEUTRINO MASS IN THE SIMPLEST EXTENSION OF THE SM

The leptonic content in the theory includes three generations of left-handed SM $SU(2)_L$ doublets and charged singlets, plus n right-handed SM singlets:

$$L_{aL} = \begin{pmatrix} \nu_a \\ l_a \end{pmatrix}_L, \quad l_{aR}, \quad \text{and } N_{bR}, \tag{11}$$

where $a = 1, 2, 3$ and $b = 1, 2, 3, \cdots, n$ ($n \geq 2$ for at least two massive neutrinos).

The leptonically universal gauge interactions involving neutrinos are of the form

$$-\mathcal{L} = \left(\frac{g}{\sqrt{2}} W^+_\mu \sum_{a=1}^{3} \overline{\nu_{aL}}\ \gamma^\mu l_{aL} + h.c. \right) + \frac{g}{2\cos_W} Z_\mu \sum_{a=1}^{3} \overline{\nu_{aL}}\ \gamma^\mu \nu_{aL}. \tag{12}$$

The gauge-invariant Yukawa interactions are

$$-\mathcal{L}_Y = \left(\sum_{a,b=1}^{3} f^l_{ab}\ \overline{L_{aL}}\ H l_{bR} + \sum_{a=1}^{3} \sum_{b=1}^{n}\ f^\nu_{ab}\ \overline{L_{aL}}\ \hat{H} N_{bR} \right) + h.c. \tag{13}$$

where H is the SM Higgs doublet

$$H = \begin{pmatrix} H^+ \\ H^0 \end{pmatrix} = \begin{pmatrix} \phi^+ \\ (h + i\phi^0)/\sqrt{2} \end{pmatrix} \tag{14}$$

and $\hat{H} = i\tau_2 H^*$. After the Higgs field develops a vev $\langle H \rangle \to v/\sqrt{2}$, the Yukawa interactions lead to Dirac masses for the leptons

$$-\mathcal{L}^D_m = \left(\sum_{a,b=1}^{3} \overline{l_{aL}}\ m^l_{ab}\ l_{bR} + \sum_{a=1}^{3} \sum_{b=1}^{n}\ \overline{\nu_{aL}}\ m^\nu_{ab}\ N_{bR} \right) + h.c. \tag{15}$$

where the mass matrices are given by the vev times the corresponding Yukawa couplings $m^{l,\nu}_{ab} = f^{l,\nu}_{ab} v/\sqrt{2}$.

The 3×3 mass matrix m^l can be diagonalized by two unitary rotations among the gauge interaction eigenstates $l_L,\ l_R$

$$O^\dagger_L\ m^l\ O_R = \text{diag}(m_e, m_\mu, m_\tau), \quad l_a = O_{a\ell}\ \ell, \tag{16}$$

where $\ell = e, \mu, \tau$ are the mass eigenstates, which define the charged lepton flavors. The Dirac masses as well as interactions with the Higgs boson for the charged leptons now have the standard form

$$-\mathcal{L}_Y^\ell = \sum_{\ell=e}^{\tau} m_\ell \ (1 + \frac{h}{v}) \ \bar{\ell} \ \ell. \tag{17}$$

If the Yukawa interactions of Eq. (13) are the whole source for neutrino mass, then we would have $\min(n, 3)$ massive Dirac neutrinos. If the Dirac neutrinos are the whole story, then the neutrino coupling to the Higgs boson would take the same form as above, directly proportional to its mass $(m_\nu/v)h\bar{\nu}\nu$.

To complete the neutrino mass sector, there is also a possible heavy Majorana mass term

$$-\mathcal{L}_m^M = \frac{1}{2} \sum_{b,b'=1}^{n} \overline{N_{bL}^c} \ B_{bb'} \ N_{b'R} + h.c. \tag{18}$$

where a charge conjugate state is defined as $\psi^c = C\bar{\psi}^T$ ($\overline{\psi^c} = \psi^T C$), and a chiral state satisfies $(\psi^c)_\tau = (\psi_{-\tau})^c$, with $\tau = L, R$. The full neutrino mass terms thus read

$$\begin{aligned} -\mathcal{L}_m^\nu &= \frac{1}{2} \left(\sum_{a=1}^{3} \sum_{b=1}^{n} (\overline{\nu_{aL}} \ m_{ab}^\nu \ N_{bR} + \overline{N_{bL}^c} \ m_{ba}^\nu \ \nu_{aR}^c) + \sum_{b,b'=1}^{n} \ \overline{N_{bL}^c} \ B_{bb'} \ N_{b'R} \right) + h.c. \\ &= \frac{1}{2} \left(\overline{\nu_L} \ \ \overline{N_L^c}\right) \begin{pmatrix} 0_{3\times3} & m_{3\times n}^\nu \\ m_{n\times3}^{\nu T} & B_{n\times n} \end{pmatrix} \begin{pmatrix} \nu_R^c \\ N_R \end{pmatrix} + h.c. \end{aligned} \tag{19}$$

where we have used the identity $\overline{\nu_{aL}} \ m_{ab} \ N_{bR} = \overline{N_{bL}^c} \ m_{ba} \ \nu_{aR}^c$,

The mass matrix can be diagonalized by one unitary transformation

$$\mathbb{L}^\dagger \begin{pmatrix} 0 & m^\nu \\ m^{\nu T} & B \end{pmatrix} \mathbb{L}^* = \begin{pmatrix} m_{diag}^\nu & 0 \\ 0 & M_{diag}^N \end{pmatrix} \tag{20}$$

where the mass eigenvalues are of the order

$$m_{diag}^\nu \approx \frac{m_\nu^2}{B}, \quad M_{diag}^N \approx B. \tag{21}$$

$\mathbb{L}$ is a $(3+n) \times (3+n)$ unitary matrix and can be parameterized as

$$\mathbb{L} = \begin{pmatrix} U_{3\times3} & V_{3\times n} \\ X_{n\times3} & Y_{n\times n} \end{pmatrix}. \tag{22}$$

The relation between the gauge interaction eigenstates and the mass eigenstates are given by

$$\begin{pmatrix} \nu_L \\ N_L^c \end{pmatrix} = \mathbb{L} \begin{pmatrix} \nu_L \\ N_L^c \end{pmatrix}_m, \tag{23}$$

with the mass eigenstates ν_m $(m = 1, 2, 3)$, $N_{m'}$ $(m' = 4, \cdots, 3+n)$. The diagonalized (Majorana) mass terms of Eq. (19) thus read

$$-\mathcal{L}_m^{\nu} = \frac{1}{2}\left(\sum_{m=1}^{3} m_m^{\nu}\ \overline{\nu_{mL}}\ \nu_{mR}^c + \sum_{m'=4}^{3+n} M_{m'}^N\ \overline{N_{m'L}^c}\ N_{m'R}\right) + h.c.\ , \tag{24}$$

with the mixing relations between the gauge and mass eigenstates

$$\nu_{aL} = \sum_{m=1}^{3} U_{am}\nu_{mL} + \sum_{m'=4}^{3+n} V_{am'}N_{m'L}^c, \quad N_{bL}^c = \sum_{m=1}^{3} X_{bm}\nu_{mL} + \sum_{m'=4}^{3+n} Y_{bm'}N_{m'L}^c, \tag{25}$$

$$\nu_{aR}^c = \sum_{m=1}^{3} U_{am}^*\nu_{mR}^c + \sum_{m'=4}^{3+n} V_{am'}^*N_{m'R}, \quad N_{bR} = \sum_{m=1}^{3} X_{bm}^*\nu_{mR}^c + \sum_{m'=4}^{3+n} Y_{bm'}^*N_{m'R}. \tag{26}$$

Note that the unitarity condition for $\mathbb{L}$ leads to the relations

$$UU^\dagger + VV^\dagger = U^\dagger U + X^\dagger X = I_{3\times 3}, \tag{27}$$

$$XX^\dagger + YY^\dagger = V^\dagger V + Y^\dagger Y = I_{n\times n}. \tag{28}$$

Parametrically, $UU^\dagger$ and $X^\dagger X \sim \mathcal{O}(1)$, $VV^\dagger$ and $Y^\dagger Y \sim \mathcal{O}(m_\nu/M_N)$.

In terms of the mass eigenstates, the gauge interaction lagrangian Eq. (12) now can be written as

$$\begin{aligned} -\mathcal{L} \ =\ & \frac{g}{\sqrt{2}}W_\mu^+\left(\sum_{\ell=e}^{\tau}\sum_{m=1}^{3}(U^\dagger O_L)_{m\ell}\ \overline{\nu_m}\gamma^\mu P_L\ell + \sum_{\ell=e}^{\tau}\sum_{m'=4}^{3+n}(V^\dagger O_L)_{m'\ell}\ \overline{N_{m'}^c}\gamma^\mu P_L\ell\right) + h.c. \\ & + \frac{g}{2\cos_W}Z_\mu\left(\sum_{m_1,m_2=1}^{3}(U^\dagger U)_{m_1m_2}\ \overline{\nu_{m_1}}\gamma^\mu P_L\nu_{m_2} + \sum_{m_1',m_2'=4}^{3+n}(V^\dagger V)_{m_1'm_2'}\overline{N_{m_1'}^c}\gamma^\mu P_L N_{m_2'}^c\right) \\ & + \frac{g}{2\cos_W}Z_\mu\left(\sum_{m_1=1}^{3}\sum_{m_2'=4}^{3+n}(U^\dagger V)_{m_1,m_2'}\overline{\nu_{m_1}}\gamma^\mu P_L N_{m_2'}^c + h.c.\right). \end{aligned} \tag{29}$$

To make the couplings more intuitive, we define the combination matrices by

$$U^{l\nu} = O_L^\dagger U, \quad V^{lN} = O_L^\dagger V, \quad U^{\nu N} = U^\dagger V, \quad U^{\nu\nu} = U^\dagger U, \quad V^{NN} = V^\dagger V. \tag{30}$$

We thus rewrite the gauge interaction lagrangian by one mixing matrix for each term

$$\begin{aligned} -\mathcal{L} \ =\ & \frac{g}{\sqrt{2}}W_\mu^+\left(\sum_{\ell=e}^{\tau}\sum_{m=1}^{3}U_{\ell m}^{l\nu *}\ \overline{\nu_m}\gamma^\mu P_L\ell + \sum_{\ell=e}^{\tau}\sum_{m'=4}^{3+n}V_{\ell m'}^{lN*}\ \overline{N_{m'}^c}\gamma^\mu P_L\ell\right) + h.c. \\ & + \frac{g}{2\cos_W}Z_\mu\left(\sum_{m_1,m_2=1}^{3}U_{m_1m_2}^{\nu\nu}\ \overline{\nu_{m_1}}\gamma^\mu P_L\nu_{m_2} + \sum_{m_1',m_2'=4}^{3+n}V_{m_1'm_2'}^{NN}\ \overline{N_{m_1'}}\gamma^\mu P_L N_{m_2'}\right) \\ & + \frac{g}{2\cos_W}Z_\mu\left(\sum_{m_1=1}^{3}\sum_{m_2'=4}^{3+n}U_{m_1m_2'}^{\nu N}\ \overline{\nu_{m_1}}\gamma^\mu P_L N_{m_2'}^c + h.c.\right). \end{aligned} \tag{31}$$

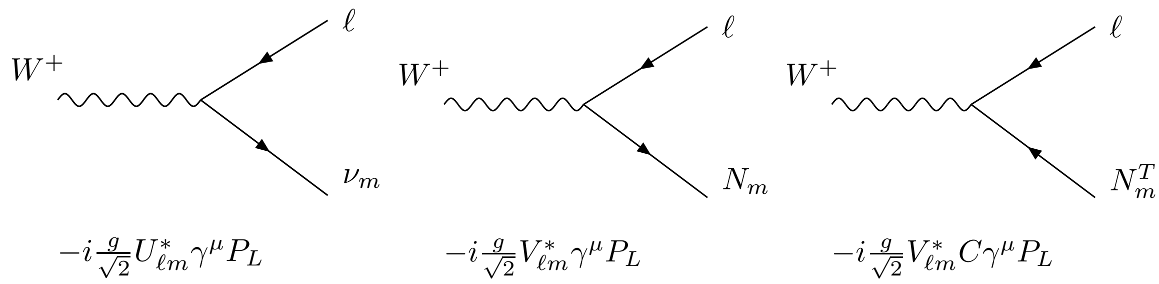

FIG. 2: Feynman rules for the charged current vertices in terms of the neutrino mass eigenstates, as given in Eq. (34).

These couplings along with the mixing matrices Eq. (30) give the most general leptonic interactions of the charged and neutral currents in terms of the mass eigenstates. Alternatively, the neutral current interactions can be aligned along with that of the charged currents when rotating left-handed neutrinos in the same way as the charged leptons,

$$\nu_{aL} = (O_L)_{a\ell}\ \nu_{\ell L},\ \text{or}\ \nu_{\ell L}\ =\ \sum_{m=1}^{3}(O_L^\dagger U)_{\ell m}\nu_{mL} + \sum_{m'=4}^{3+n}(O_L^\dagger V)_{\ell m'}N^c_{m'L}. \tag{32}$$

It may be convenient in certain practical calculations to rewrite the neutral current interactions in terms of their flavor eigenstates

$$-\mathcal{L}\ =\ \frac{g}{\sqrt{2}}W_\mu^+\left(\sum_{\ell=e}^{\tau}\sum_{m=1}^{3}U^*_{\ell m}\ \overline{\nu_m}\gamma^\mu P_L\ell + \sum_{\ell=e}^{\tau}\sum_{m'=4}^{3+n}V^*_{\ell m'}\ \overline{N^c_{m'}}\gamma^\mu P_L\ell\right) + h.c. \tag{33}$$

$$+\ \frac{g}{2\cos\theta_W}Z_\mu\left(\sum_{\ell=e}^{\tau}\sum_{m=1}^{3}U^*_{\ell m}\ \overline{\nu_m}\gamma^\mu P_L\ \nu_\ell + \sum_{\ell=e}^{\tau}\sum_{m'=4}^{3+n}V^*_{\ell m'}\ \overline{N^c_{m'}}\gamma^\mu P_L\ \nu_\ell\right) + h.c. + ... \tag{34}$$

where we have dropped the superscripts for U, V defined in Eq. (30), for simplicity as we adopted throughout the text.

For the reader's convenience, we give most of the corresponding Feynman rules for the interaction vertices, listed in Fig. 2 for the charged currents, and in Fig. 3 for the neutral currents. The Feynman rules for the other diagrams can be easily deduced from the ones that are explicitly written down in Fig. 2 and Fig. 3.

Finally, the heavy neutrino interactions with the Higgs boson read

$$-\mathcal{L}_{\mathcal{H}} = \frac{H}{v}\sum_{\ell=e}^{\tau}\sum_{m'=4}^{3+n}V^*_{\ell m'}\ M^N_{m'}\ \overline{N^c_{m'}}P_L\ \nu_\ell + h.c. + ... \tag{35}$$

The corresponding Feynman rule for the interaction vertex is given in Fig. 4.

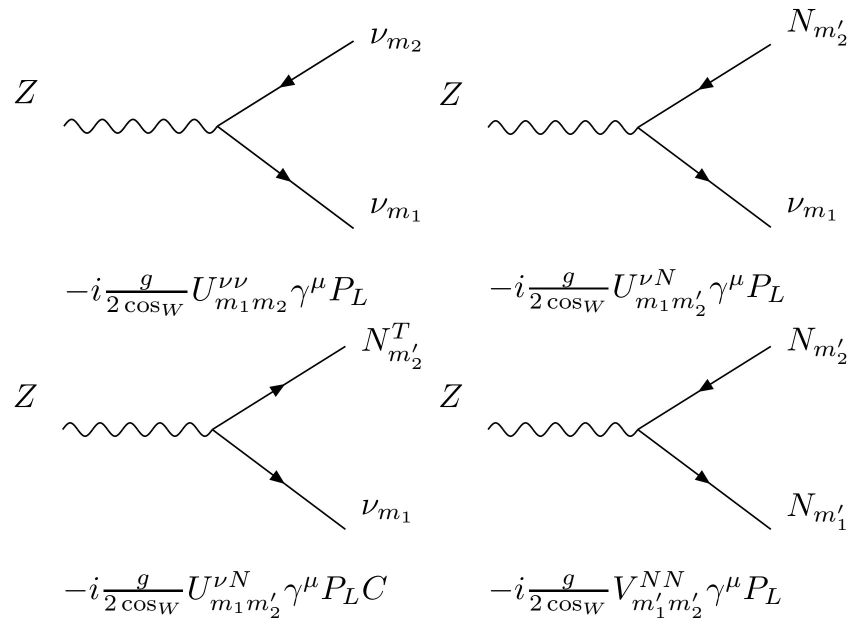

FIG. 3: Feynman rules for the neutral current vertices in terms of the neutrino mass eigenstates, as given in Eqs. (31) and (34).

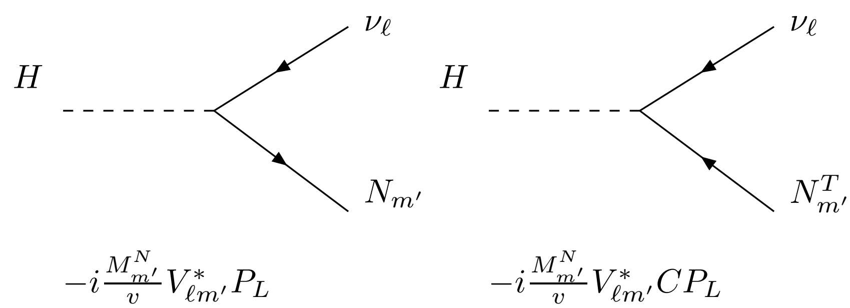

FIG. 4: Feynman rule for the Higgs vertex in terms of the heavy neutrino mass eigenstate, as given in Eq. (35).

II. GENERAL AMPLITUDE OF $\Delta L = 2$ PROCESSES

The charged current interaction Lagrangian in terms of neutrino mass eigenstates is

$$\mathcal{L}_{cc} = -\frac{g}{\sqrt{2}}W^+_\mu\Big(\sum_{\ell=e}^{\tau}\sum_{m=1}^{3}U^{l\nu *}_{\ell m}\,\overline{\nu_m}\gamma^\mu P_L\ell + \sum_{\ell=e}^{\tau}\sum_{m'=4}^{3+n}V^{lN*}_{\ell m'}\,\overline{N^c_{m'}}\gamma^\mu P_L\ell\Big) + \text{h.c.} \tag{36}$$

where $P_L = \frac{1}{2}(1-\gamma_5)$.

The basic process with $\Delta L = 2$ can be generically expressed by

$$W^-W^- \to \ell^-_1\ell^-_2, \tag{37}$$

where W^- is a virtual SM weak boson and $\ell_{1,2} = e, \mu, \tau$. The leptonic $\Delta L = 2$ subprocess is induced by the product of two charged currents

$$\mathcal{M}^{\mu\nu}_{lep} \propto \sum_{m=1}^{3} U^{l\nu}_{\ell_1 m} U^{l\nu}_{\ell_2 m} \; (\overline{\ell_1}\gamma^\mu P_L \nu_m)(\overline{\ell_2}\gamma^\nu P_L \nu_m) + \sum_{m'=4}^{3+n} V^{lN}_{\ell_1 m'} V^{lN}_{\ell_2 m'} \; (\overline{\ell_1}\gamma^\mu P_L N_{m'})(\overline{\ell_2}\gamma^\nu P_L N_{m'}), \tag{38}$$

which can be rewritten using charge conjugation as

$$\mathcal{M}^{\mu\nu}_{lep} \propto \sum_{m=1}^{3} U^{l\nu}_{\ell_1 m} U^{l\nu}_{\ell_2 m} \; (\overline{\ell_1}\gamma^\mu P_L \nu_m)(\overline{\nu_m}\gamma^\nu P_R \ell_2^c) + \sum_{m'=4}^{3+n} V^{lN}_{\ell_1 m'} V^{lN}_{\ell_2 m'} \; (\overline{\ell_1}\gamma^\mu P_L N_{m'})(\overline{N_{m'}}\gamma^\nu P_R \ell_2^c). \tag{39}$$

The Majorana neutrino fields can be contracted to form a neutrino propagator, and the transition matrix element is thus given by

$$\begin{aligned} \mathcal{M}^{\mu\nu}_{lep} &= \frac{g^2}{2} \sum_{m=1}^{3} U^{l\nu}_{\ell_1 m} U^{l\nu}_{\ell_2 m} \; (\overline{\ell_1}\gamma^\mu P_L) \frac{\not{q} + m_{\nu_m}}{q^2 - m^2_{\nu_m} + i\Gamma_{\nu_m} m_{\nu_m}} (\gamma^\nu P_R \ell_2^c) \\ &+ \frac{g^2}{2} \sum_{m'=4}^{3+n} V^{lN}_{\ell_1 m'} V^{lN}_{\ell_2 m'} \; (\overline{\ell_1}\gamma^\mu P_L) \frac{\not{q} + m_{N_{m'}}}{q^2 - m^2_{N_{m'}} + i\Gamma_{N_{m'}} m_{N_{m'}}} (\gamma^\nu P_R \ell_2^c), \end{aligned} \tag{40}$$

where q is the momentum exchange carried by the neutrino. The $\not{q}$ term vanishes due to the chirality flip. Including the crossed diagram ($\ell_1 \leftrightarrow \ell_2$) the leptonic amplitude then becomes

$$\begin{aligned} \mathcal{M}^{\mu\nu}_{lep} &= \frac{g^2}{2} \sum_{m=1}^{3} U^{l\nu}_{\ell_1 m} U^{l\nu}_{\ell_2 m} \; m_{\nu_m} \overline{u_1} \left(\frac{\gamma^\mu \gamma^\nu}{q^2 - m^2_{\nu_m} + i\Gamma_{\nu_m} m_{\nu_m}} + \frac{\gamma^\nu \gamma^\mu}{q'^2 - m^2_{\nu_m} + i\Gamma_{\nu_m} m_{\nu_m}} \right) P_R v_2 \\ &+ \frac{g^2}{2} \sum_{m'=4}^{3+n} V^{lN}_{\ell_1 m'} V^{lN}_{\ell_2 m'} \; m_{N_{m'}} \times \\ &\quad \overline{u_1} \left(\frac{\gamma^\mu \gamma^\nu}{q^2 - m^2_{N_{m'}} + i\Gamma_{N_{m'}} m_{N_{m'}}} + \frac{\gamma^\nu \gamma^\mu}{q'^2 - m^2_{N_{m'}} + i\Gamma_{N_{m'}} m_{N_{m'}}} \right) P_R v_2. \end{aligned} \tag{41}$$

For the light Majorana neutrinos, namely, $m = 1, 2, 3$ the masses $m_{\nu_m} \sim \mathcal{O}(\text{eV})$ and for the heavy Majorana neutrinos , the masses $m_{N_{m'}} \sim \mathcal{O}(\text{MeV} - \text{GeV})$ for the low energy processes we consider. The heavy Majorana neutrino contribution has a resonant enhancement when $q^2, q'^2 \approx m^2_{N_{m'}}$ and is the dominant one. We can neglect the contributions of the light Majorana neutrinos and the $\sum_{m=1}^{3}$ part of the amplitude drops out.

In principle all the heavy Majorana neutrinos will contribute to the amplitude but in our analysis we only consider the contribution of one of the heavy neutrinos, in particular the lightest one. This is a reasonable assumption for the following reasons. The transition rate in the resonantly enhanced case is proportional to the square of the mixing element $V_{\ell m'}$.

The mixing elements are of the order of magnitude (m^ν/B) and in general grow smaller with increasing mass of the heavy neutrino. So it is reasonble to study the contribution of the lightest heavy neutrino which corresponds to the largest mixing. The contribution from the other heavier neutrinos will be suppressed accordingly. So the amplitude can now be written as

$$\mathcal{M}^{\mu\nu}_{lep} = \frac{g^2}{2} V_{\ell_1 4} V_{\ell_2 4}\ m_4 \overline{u_1} \left(\frac{\gamma^\mu \gamma^\nu}{q^2 - m_4^2 + i\Gamma_{N_4} m_4} + \frac{\gamma^\nu \gamma^\mu}{q'^2 - m_4^2 + i\Gamma_{N_4} m_4} \right) P_R v_2. \tag{42}$$

We can rewrite the above amplitude as

$$\begin{aligned} \mathcal{M}^{\mu\nu}_{lep} &= \frac{g^2}{2} V_{\ell_1 4} V_{\ell_2 4}\ m_4 \frac{\overline{u_1}\gamma^\mu \gamma^\nu P_R v_2}{q^2 - m_4^2 + i\Gamma_{N_4} m_4} + \frac{g^2}{2} V_{\ell_1 4} V_{\ell_2 4}\ m_4 \frac{\overline{u_1}\gamma^\nu \gamma^\mu P_R v_2}{q'^2 - m_4^2 + i\Gamma_{N_4} m_4} \\ &= \mathcal{M}_1 + \mathcal{M}_2. \end{aligned} \tag{43}$$

When $q^2 \approx m_4^2$, $\mathcal{M}_1$ has a resonant contribution and when $q'^2 \approx m_4^2$, $\mathcal{M}_2$ has a resonant contribution.

The dynamics for $\Delta L = 2$ processes as in Eq. (37) is dictated by the properties of the exchanged neutrinos. For a Majorana neutrino that is light compared to the energy scale in the process, the transition rates for LV processes are proportional to the product of two flavor mixing matrix elements among the light neutrinos and a LV mass insertion

$$\langle m \rangle^2_{\ell_1 \ell_2} = \left| \sum_{m=1}^{3} U_{\ell_1 m} U_{\ell_2 m} m_{\nu_m} \right|^2, \tag{44}$$

where $\langle m \rangle_{\ell_1 \ell_2}$ are the "effective neutrino masses". If the neutrinos are heavy compared to the energy scale involved, then the contribution scales as

$$\left| \sum_{m'=4}^{3+n} \frac{V_{\ell_1 m'} V_{\ell_2 m'}}{m_{N_{m'}}} \right|^2, \tag{45}$$

where V is the mixing matrix between the light flavor and heavy neutrinos. Unfortunately, both situations encounter a severe suppression either due to the small neutrino mass like $m^2_{\nu_m}/M_W^2$, or due to the small mixing $|V_{\ell_1 m'} V_{\ell_2 m'}|^2$.

The full amplitude for a $\Delta L = 2$ process as depicted by Fig. 1 can be casted into the leptonic part and the fermionic currents via the W-exchange

$$i\mathcal{M} = (\mathcal{M}_{lep})_{\mu\nu} J^\mu(f_1' f_1) J^\nu(f_2' f_2). \tag{46}$$

One sees that by coupling fermion currents to the W bosons as depicted in Fig. 1, and arranging the initial and final states properly, one finds various physical processes that can be experimentally searched for. The best known example is the neutrinoless double-beta decay ($0\nu\beta\beta$), which proceeds via the parton-level subprocess $dd \to uu\ W^{-*}W^{-*} \to uu\ e^-e^-$.

An important observation is that when the heavy neutrino mass is kinematically accessible, a process may undergo resonant production of the heavy neutrino in an intermediate s-channel. The transition rate can be substantially enhanced and goes like

$$\frac{\Gamma(N_{m'} \to i)\ \Gamma(N_{m'} \to f)}{m_{N_{m'}}\Gamma_{N_{m'}}}, \tag{47}$$

where i, f refer to the initial and final state during the transition.

One of the interesting process is the tau decay such as $\tau^- \to \ell^+ M_1^- M_2^-$ where the light mesons M_1, M_2 are π, K. To construct the decay amplitude, we first note that the leptonic part of the subprocess $\tau^- \to \ell^+ W^{-*}W^{-*}$ is obtained by crossing the amplitude

$$\mathcal{M}^{\mu\nu}_{lep} = \frac{g^2}{2} V^*_{\tau 4} V^*_{\ell 4}\ \overline{v_\tau} \frac{m_4}{q^2 - m_4^2 + i\Gamma_{N_4} m_4} \gamma^\mu \gamma^\nu P_R v_\ell. \tag{48}$$

Combining the hadronic and leptonic parts, the decay amplitude for

$$\tau^-(p_1) \to \ell^+(p_2)\ M_1^-(q_1)\ M_2^-(q_2) \tag{49}$$

is given by

$$\begin{aligned} i\mathcal{M} &= (\mathcal{M}_{lep})_{\mu\nu} \mathcal{M}^\mu_{M_1} \mathcal{M}^\nu_{M_2} + (M_1 \leftrightarrow M_2) && (50)\\ &= 2G_F^2 V^{CKM}_{M_1} V^{CKM}_{M_2} f_{M_1} f_{M_2} V^*_{\tau 4} V^*_{\ell 4}\ m_4 \left[\frac{\overline{v_\tau}\ \not{q}_1\ \not{q}_2 P_R v_\ell}{(p_1 - q_1)^2 - m_4^2 + i\Gamma_{N_4} m_4} \right] \\ &+ 2G_F^2 V^{CKM}_{M_1} V^{CKM}_{M_2} f_{M_1} f_{M_2} V^*_{\tau 4} V^*_{\ell 4}\ m_4 \left[\frac{\overline{v_\tau}\ \not{q}_2\ \not{q}_1 P_R v_\ell}{(p_1 - q_2)^2 - m_4^2 + i\Gamma_{N_4} m_4} \right], && (51)\\ &= \mathcal{M}_1 + \mathcal{M}_2, && (52) \end{aligned}$$

where $V^{CKM}_{M_i}$ are the quark flavor-mixing matrix elements for the mesons and f_{M_i} are meson decay constants.

Other interesting processes may include rare meson decays such as $M_1^+ \to \ell_1^+ \ell_2^+ M_2^-$ and hyperon decays such as $\Sigma^- \to \Sigma^+ e^- e^-$, $\Xi^- \to p\mu^-\mu^-$ etc. One could also explore additional processes like $e^- \to \mu^+$ and $\mu^- \to e^+$ conversion. One may also consider searching for signals at accelerator and collider experiments via $e^-e^- \to W^-W^-$ $e^+e^- \to Z^0 \to N + X$ $e^\pm p \to \nu_e(\overline{\nu_e})\ell_1^\pm \ell_2^\pm X$ neutrino nucleon scattering $\nu_\ell(\overline{\nu_\ell})\mathcal{N} \to \ell^\mp \ell_1^\pm \ell_2^\pm X$ and hadronic production $pp \to \ell_1^+ \ell_2^+ X$.

For a reference, see Anupama Atre, Tao Han, Silvia Pascoli, and Bin Zhang, JHEP 0905:030,2009, arXiv:0901.3589 [hep-ph]