

# Modelling Chromatic Emittance Growth in Staged Plasma Wakefield Acceleration to 1 TeV using Nonlinear Transfer Matrices<sup>1</sup>

[1] A.G.R. Thomas, D. Seipt, Phys. Rev. Accel. Beams **24**, 104602 (2021)

**Alec Thomas**

Gérard Mourou Center for Ultrafast Optical Sciences, and  
Department of Nuclear Engineering and Radiological Sciences, University of Michigan,  
Ann Arbor, MI 48109, USA

**Daniel Seipt**

Helmholtz Institut Jena, Fröbelstieg 3, 07743 Jena, and  
GSI Helmholtzzentrum für Schwerionenforschung GmbH, Planckstrasse 1, 64291  
Darmstadt, Germany

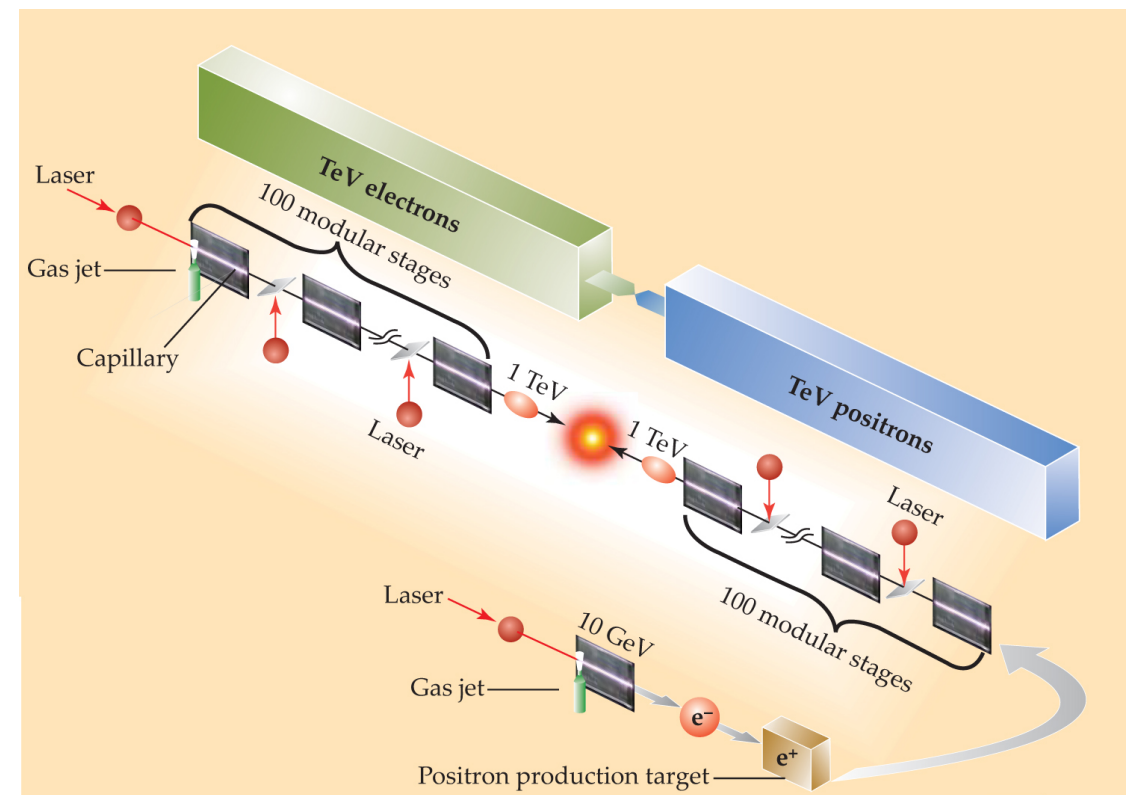
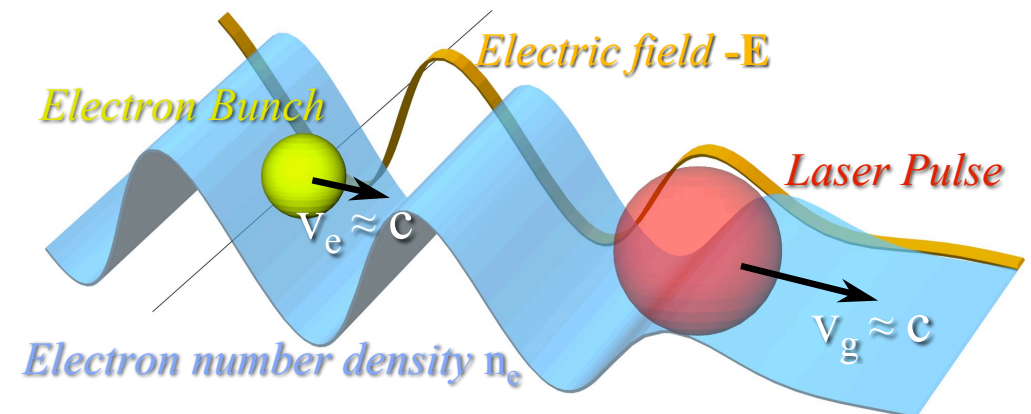
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# Towards a plasma collider

## Motivation for detailed beam dynamics studies

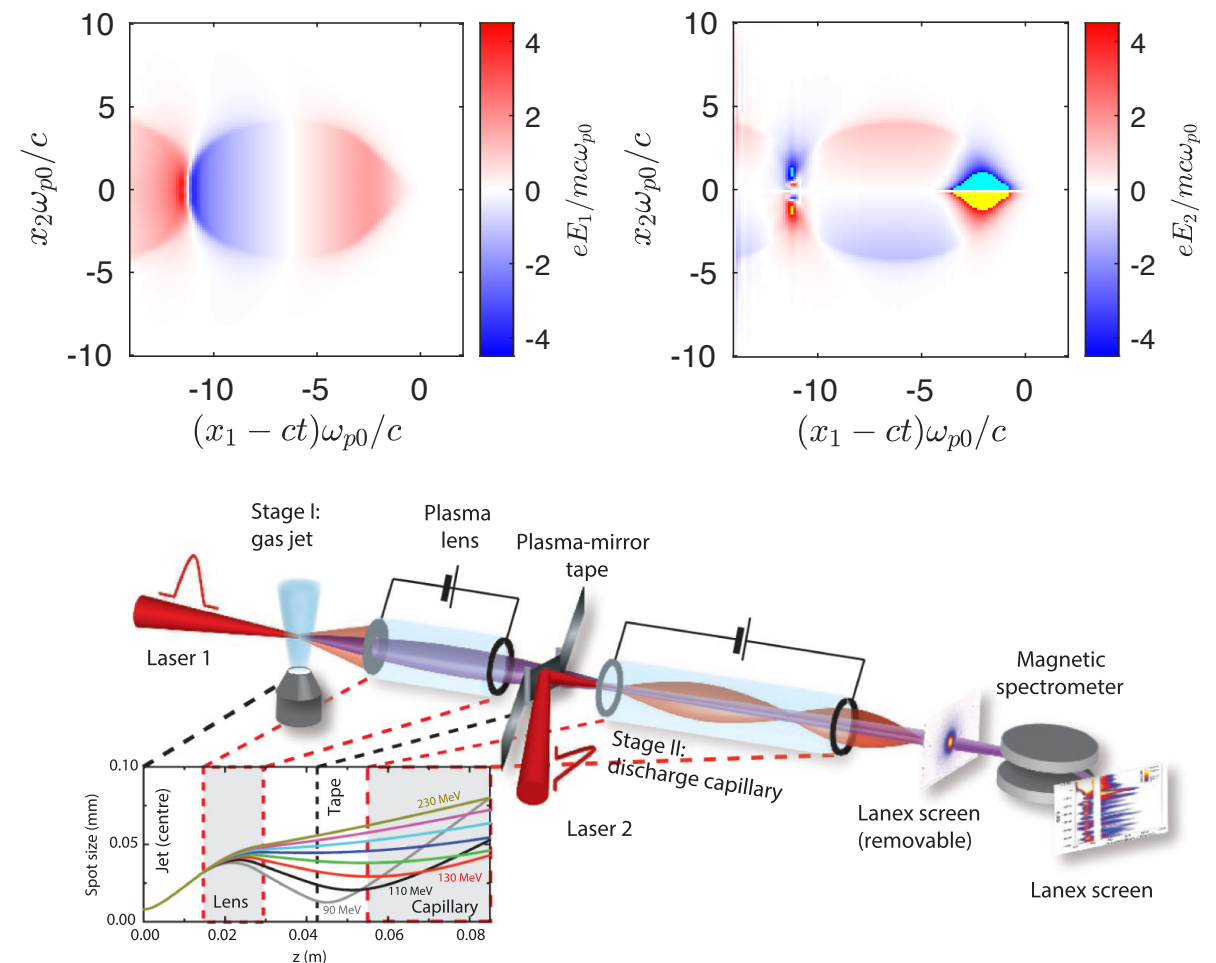
- Plasma accelerators use either a particle beam or laser pulse driver to generate a wakefield that may be used to accelerate leptons to high energy
- An application of plasma accelerators is a compact TeV lepton collider<sup>1</sup>
- No definitive scheme but staged acceleration seems likely



# Staging for plasma accelerators

## Motivation for detailed beam dynamics studies

- Wakes in 3D have **strong transverse fields** that cause particle transverse oscillations
- Coupling of two plasma stages demonstrated at BELLA<sup>1</sup>
- But staging to high energies has the issue that the strong focusing required for the micron scale beams in plasma accelerators is highly chromatic<sup>2-4</sup>
- Energy spreads in plasma accelerators have not been small in practice



- [1] S. Steinke, J. van Tilborg, C. Benedetti, C. G. R. Geddes, C. B. Schroeder, J. Daniels, K. K. Swanson, A. J. Gonsalves, K. Nakamura, N.H. Matlis et al., Nature (London) 530, 190 (2016).
- [2] T. Mehrling, J. Grebenyuk, F. S. Tsung, K. Floettmann, and J. Osterhoff, Phys. Rev. ST Accel. Beams 15, 111303 (2012).
- [3] C.A. Lindstrøm, Phys. Rev. Accel. Beams 24, 014801 (2021).
- [4] M. D. Litos, R. Ariniello, C. E. Doss, K. Hunt-Stone, and J. R. Cary, Phil. Trans. R. Soc. A 377, 20180181 (2019).



# Some considerations for staging

Not comprehensive!

PHYSICAL REVIEW ACCELERATORS AND BEAMS **24**, 014801 (2021)

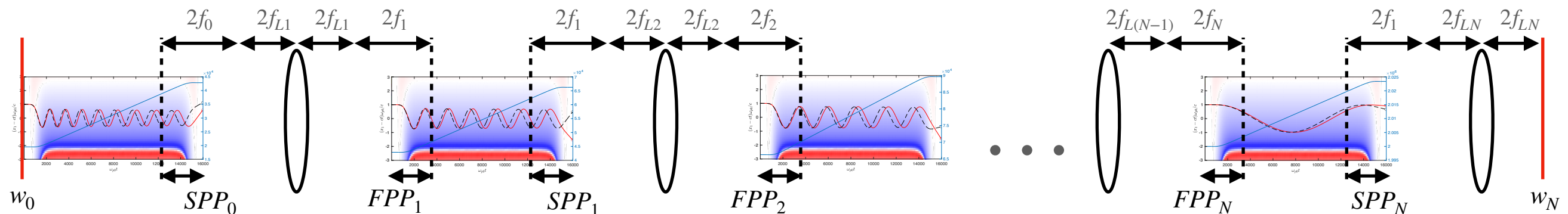
Review Article

## Staging of plasma-wakefield accelerators

Carl A. Lindström\*

Deutsches Elektronen-Synchrotron DESY, Notkestraße 85, 22607 Hamburg, Germany

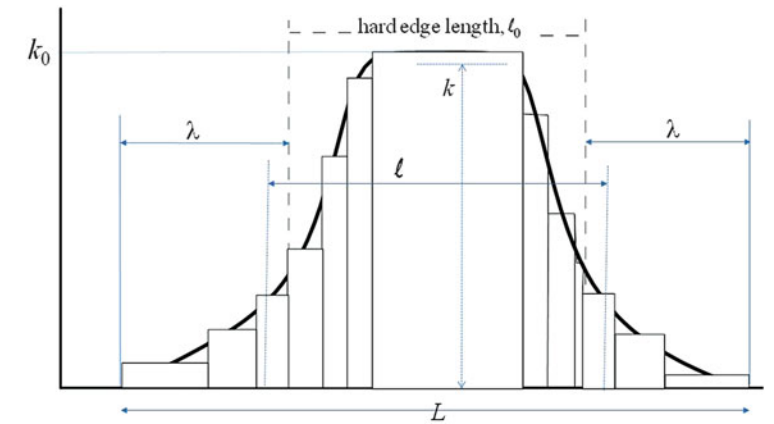
(Received 5 May 2020; accepted 4 January 2021; published 25 January 2021)



- Lattice consisting of accelerator stages along with lenses and other elements (plasma dechirpers etc.)
- Plasma accelerator itself has strong focusing fields and acts as lens
- Want to determine **stable, transverse emittance preserving, efficient (compact)** lattice design
- Want to understand constraints on witness beam properties

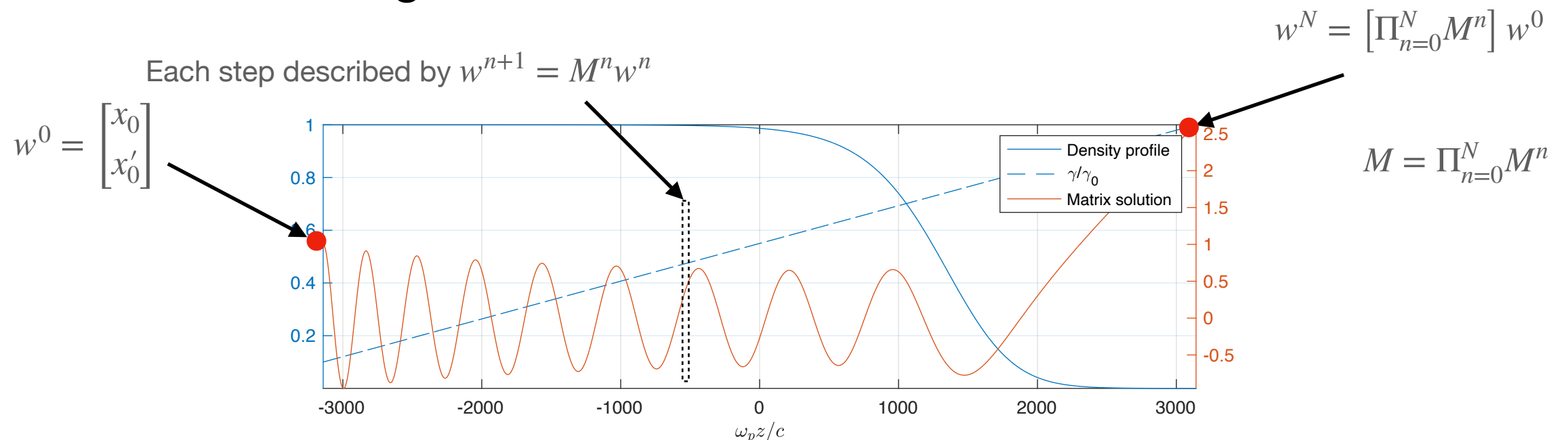
# Why transfer matrices?

## Motivation



H. Wiedemann, Particle Accelerator Physics, Graduate Texts in Physics (Springer, Berlin, Germany, 2015)

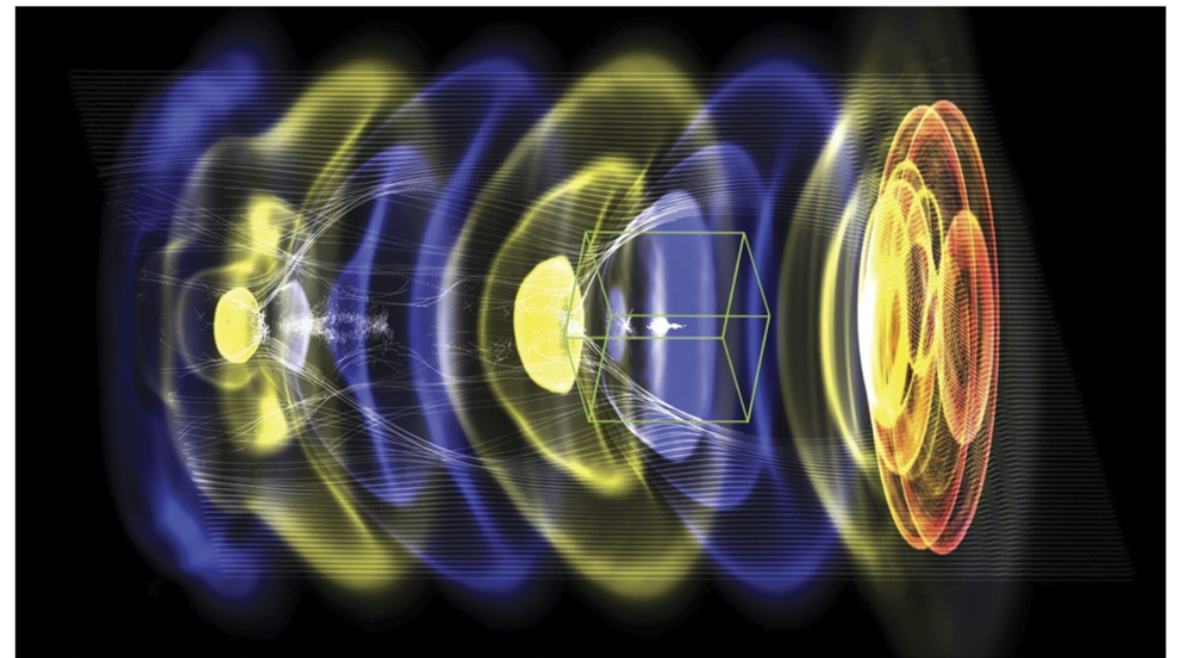
- Well developed formalism in conventional accelerators
- For numerical solution, instead of integrating single particle track, find matrix describing transformation of coordinates from  $x_0, x'_0$  to  $x, x'$
- Can use compound matrix  $M$  to calculate end states of large number of particles or infer properties of element e.g. focusing characteristics, eigenmodes of the lattice



# Why transfer matrices?

## Compared to particle-in-cell solutions

- Particle-in-cell simulations highly advanced and can accurately simulate plasma accelerator
- Computationally expensive - a single simulation of TeV scale accelerator on Exascale computer is a challenge, let alone a parameter scan
- Need a faster method for design exploration
  - To guide parameter choices for full-scale simulation



**Figure 19.** Snapshot from a WarpX 3D simulation of a laser-driven plasma accelerator. The laser (red) propagates from left to right and creates a plasma wake (yellow and blue) that accelerates a small electron beam (white) to high energy. A mesh refinement patch (green box) is used to increase the resolution and accuracy around the electron beam. Credit: Maxence Thévenet.

# Why transfer matrices?

## Relation to particle distribution

- Can relate beam matrix to averages of the particle distribution:

$$\langle x^2 \rangle = \sigma_{11} = \epsilon\beta$$

$$\langle xx' \rangle = \sigma_{12} = \sigma_{21} = -\epsilon\alpha$$

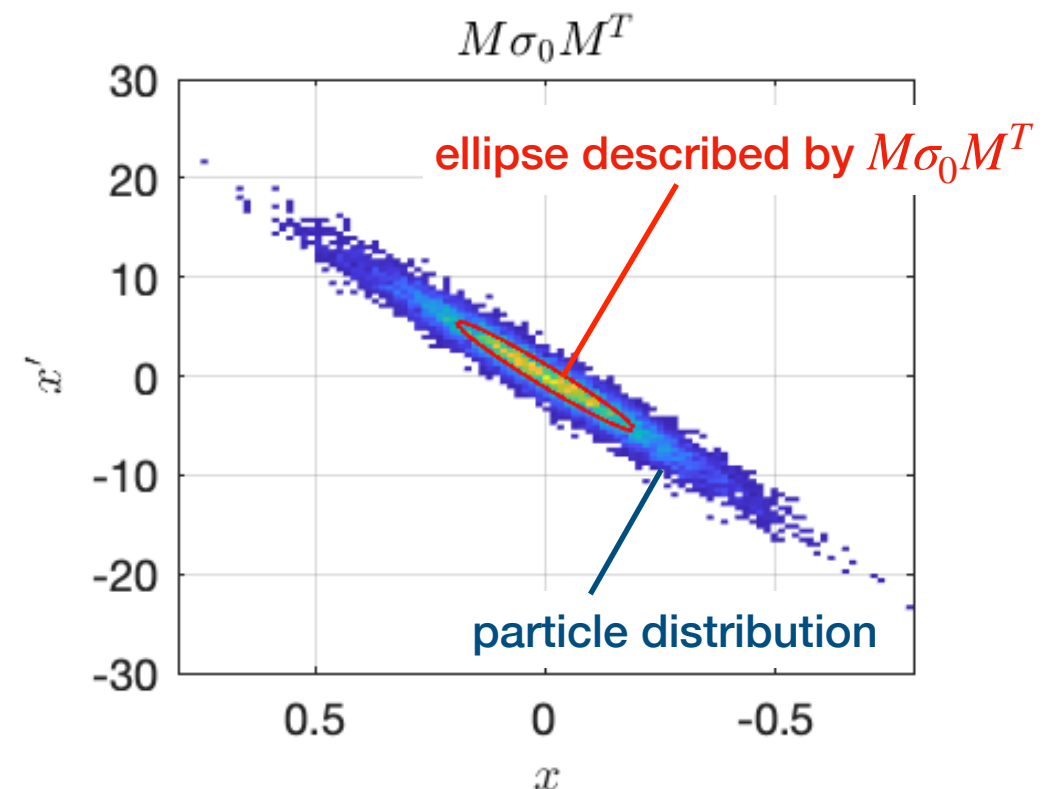
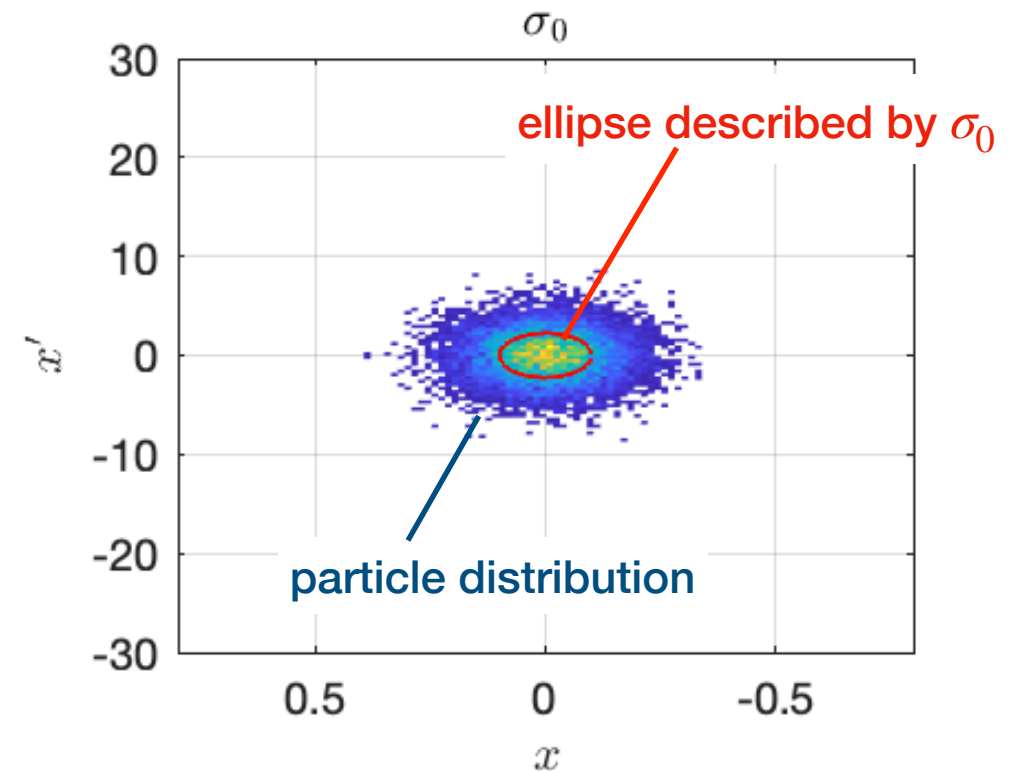
$$\langle x'^2 \rangle = \sigma_{22} = \epsilon\gamma$$

- The beam matrix transforms with the transfer matrix describing the transformation of the single particle coordinates as

$$\sigma = M\sigma M^T$$

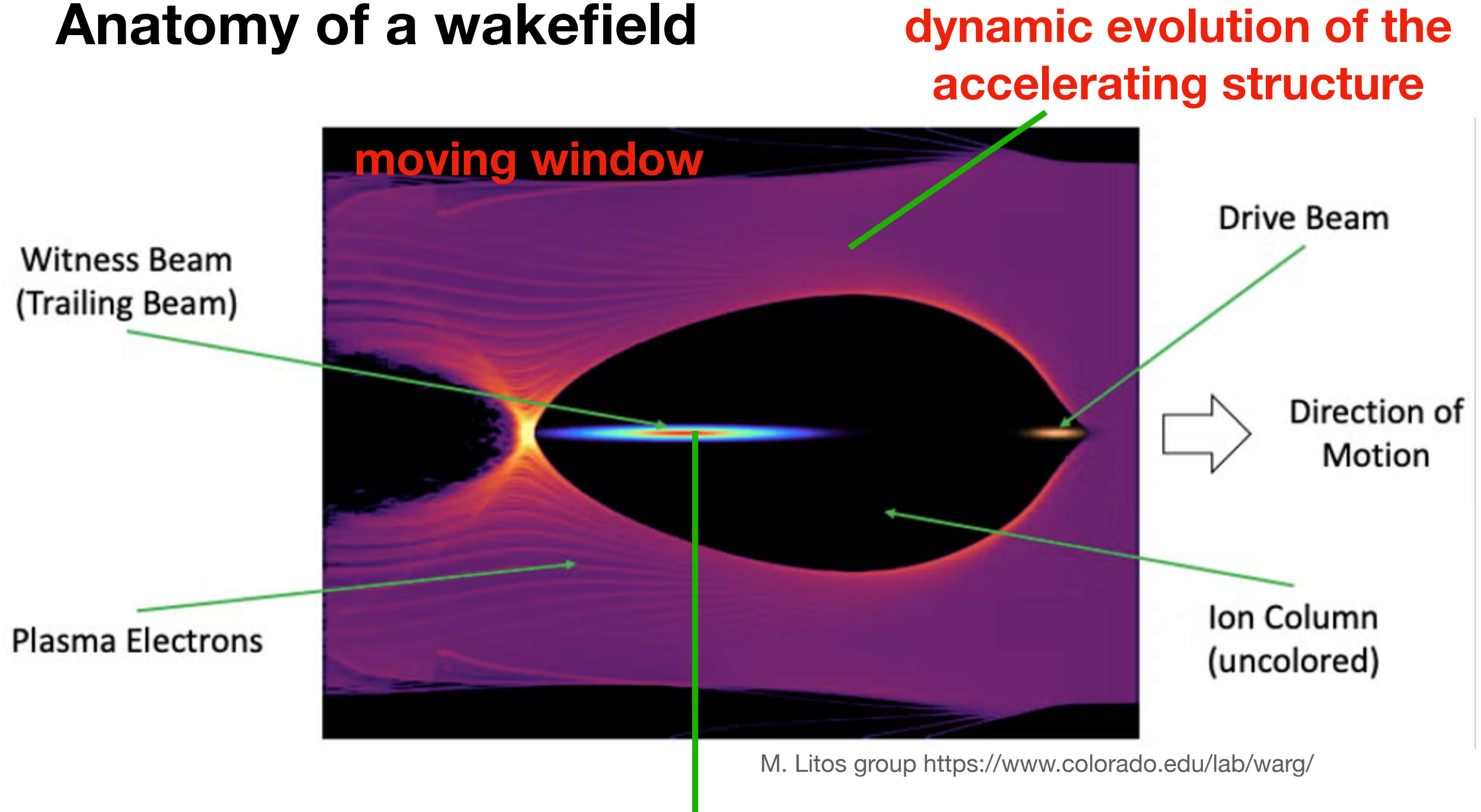
- The emittance is defined as

$$\epsilon = \sqrt{\det \sigma}$$



# Transfer matrices for plasma accelerators

## Anatomy of a wakefield



M. Litos group <https://www.colorado.edu/lab/warg/>

**Witness beam moves at close to  $c$**

# Transfer matrices for plasma accelerators

## Basic assumptions

1. ***Constant speed of light phase approximation*** we assume that the witness beam is ultrarelativistic propagating in the z direction:

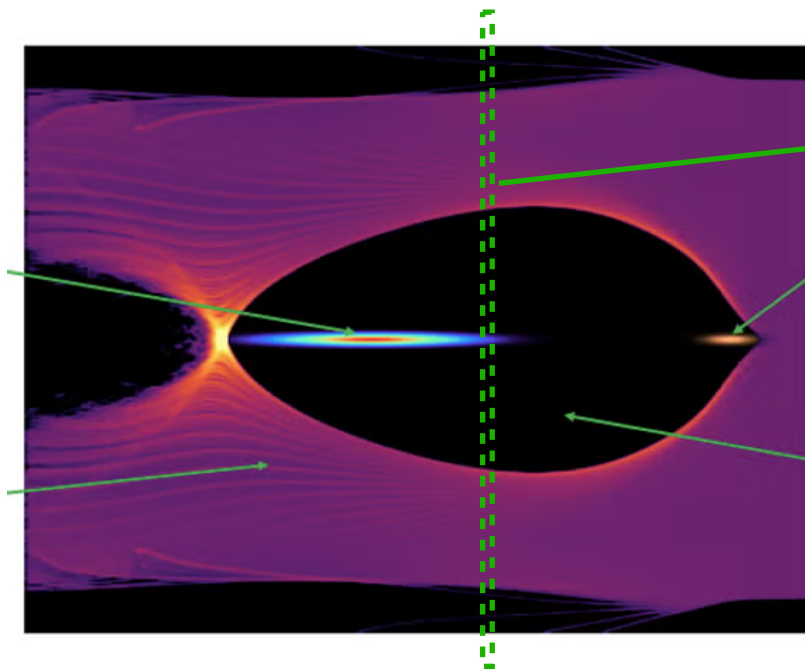
$$1 - \beta_z \lll 1$$

- In practice, for a 1 GeV electron beam,  $1 - \beta_z \sim 10^{-7}$ , so after 1 meter propagation in a stage the phase slippage relative to light speed is a tenth of a micron
- This approximation means that the beam position with respect to light speed coordinates  $\xi = z - ct$  is constant to good approximation
- Particle experiences fields varying in time only at fixed position  $\xi_0$
- i.e. from Particle-In-Cell simulation in “moving window”, we may simply record the fields at some fixed position in the window



# Transfer matrices for plasma accelerators

## Basic assumptions



Extract fields from  
simulation slice  $\xi_0$

Each slice treated as matrix  
transformation; whole  
simulation converted to  
matrix transformation

2. ***External fields approximation;*** no-self consistent fields due to beam particles, only wakefield owing to driver

- There is no feedback between the particles and the wake - cannot account for e.g. beam loading
- We model the dynamics of the “first particle” in the witness beam
- Can find stable / emittance preserving lattice conditions - then hand off to more expensive full simulation methods

# Transfer matrices for plasma accelerators

## Basic assumptions

### 3. *Paraxial beam approximation*

- The accelerated beam is close to the axis
- The fields are null on axis
- This means we may use the standard paraxial approximations

$$F_l(x, y, z) = F_l(0, 0, z) + x \frac{\partial F_l}{\partial x}(0, 0, z) + y \frac{\partial F_l}{\partial y}(0, 0, z) + \dots$$

- And  $\beta_z \rightarrow 1$  means the transverse equations of motion are

$$\frac{dx}{dt} = \frac{u_x}{\gamma}, \quad \frac{dy}{dt} = \frac{u_y}{\gamma}, \quad \frac{du_x}{dt} \simeq -\alpha_{xx}^2 x - \alpha_{xy}^2 y, \quad \frac{du_y}{dt} \simeq -\alpha_{yx}^2 x - \alpha_{yy}^2 y$$

$$\alpha_{kl}^2 = -\frac{q}{m} \frac{\partial}{\partial k} (E_l - \varepsilon_{lp} c B_p) \Big|_{x=0, y=0}$$

# Transfer matrices for plasma accelerators

## Basic assumptions

- We assume that the time step in the simulation is sufficiently small that the beam energy can be simply incremented as

$$\gamma^n \simeq \gamma_0 + \sum_{n'=0}^n \frac{qE_z^{n'} \Delta t}{mc}$$

- The transverse equations of motion can be solved over a timestep in matrix form:

$$\begin{bmatrix} x^{n+1} \\ u_x^{n+1} \\ y^{n+1} \\ u_y^{n+1} \end{bmatrix} = \begin{bmatrix} 1 - \frac{[\alpha_{xx}^2]^n \Delta t^2}{\gamma^n} & \frac{\Delta t}{\gamma^n} & -\frac{[\alpha_{xy}^2]^n \Delta t^2}{\gamma^n} & 0 \\ -[\alpha_{xx}^2]^n \Delta t & 1 & -[\alpha_{xy}^2]^n \Delta t & 0 \\ -\frac{[\alpha_{yx}^2]^n \Delta t^2}{\gamma^n} & 0 & 1 - \frac{[\alpha_{yy}^2]^n \Delta t^2}{\gamma^n} & \frac{\Delta t}{\gamma^n} \\ -[\alpha_{yx}^2]^n \Delta t & 0 & -[\alpha_{yy}^2]^n \Delta t & 1 \end{bmatrix} \begin{bmatrix} x^n \\ u_x^n \\ y^n \\ u_y^n \end{bmatrix}$$

- This is basically equivalent to a second order leapfrog pusher scheme (and therefore has desirable conservation properties; symplectic provided  $a_{xy} = a_{yx}$ , which is true if the force is curl free - i.e. conservative)

# Transfer matrices for plasma accelerators

## Basic assumptions

- The transformation of a particle's transverse coordinates through an entire plasma simulation can therefore be written as

$$M_j(\gamma_0) = \prod_{n=0}^{N_t} M_j^n(\gamma_j^n)$$

- Where  $M_j^n$  is the transfer matrix for the nth timestep at the jth position in light speed coordinate  $\xi_j$ .
- The particle coordinates therefore transform through the plasma stage as

$$w = M_j(\gamma_0)w_0$$

# Chromatic emittance growth

## Betatron orbit dispersion

- Equation of motion in x direction, no cross term:

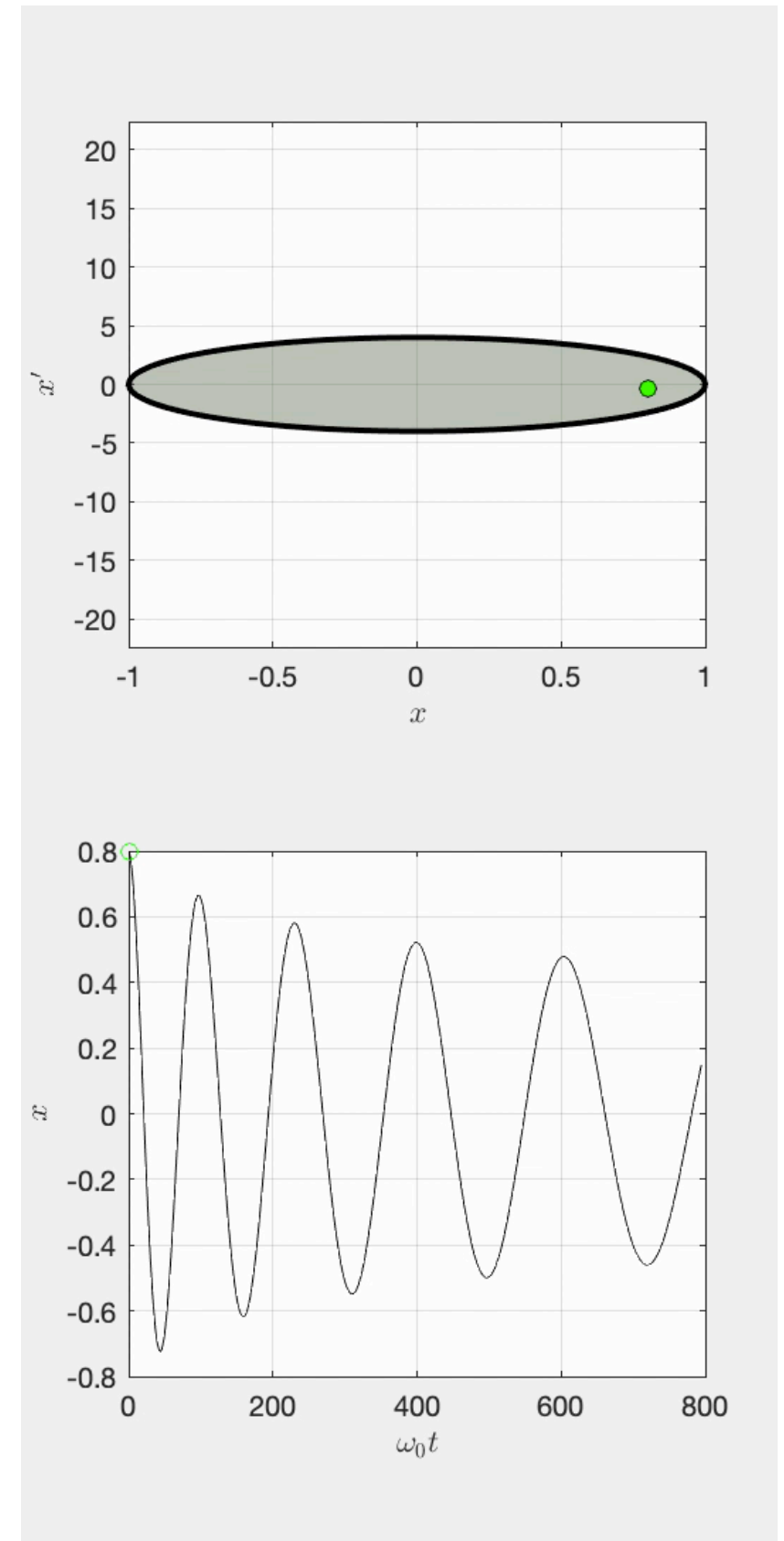
$$\frac{du_x}{dt} = -\alpha(\tau)^2 x \Rightarrow \frac{d^2 x}{d\tau^2} = -\gamma(\tau)\alpha(\tau)^2 x$$

where  $d\tau = dt/\gamma$

- WKB solution

$$x = x_0 \left( \frac{\alpha(s)\gamma_0}{\alpha_0\gamma(s)} \right)^{1/4} \cos \int_0^t \frac{\alpha}{\sqrt{\gamma(s)}} ds$$

- Particles with different energies will undergo betatron oscillations with different frequencies<sup>1</sup>
- Leads to emittance growth as phase space ellipses associated with different energies disperse



[1] T. Mehrling, J. Grebenyuk, F. S. Tsung, K. Floettmann, and J. Osterhoff, Phys. Rev. ST Accel. Beams 15, 111303 (2012).

# Chromatic emittance growth

## New nonlinear matrix formalism

$$\frac{d^2x}{dt^2} \simeq -\frac{\alpha^2}{\gamma + \delta\gamma}x \approx -\frac{\alpha^2}{\gamma} \left( 1 - \frac{\delta\gamma}{\gamma} + \frac{\delta\gamma^2}{\gamma^2} + \dots \right)$$

- To treat this problem efficiently, **we don't want to have to recalculate the matrix for each energy**
- ...use perturbation theory and expand the matrix in powers of the energy spread  $\delta\gamma$  about a **design energy**  $\gamma$ :

$$M(\gamma + \delta\gamma) = M(\gamma) + \sum_{p=1}^{\infty} (-1)^p \left( \frac{\delta\gamma}{\gamma} \right)^p M_D \quad M_D = -\gamma \left. \frac{\partial M}{\partial \gamma} \right|_{\gamma}$$

- Solve nonlinear matrix equation<sup>1</sup>

<sup>1</sup> K. L. Brown, Adv. Part. Phys. 1, 71 (1968)

$$w_{\delta} = \mathcal{M} w_{\delta 0}$$

$$w_{\delta} = \begin{bmatrix} 1 \\ \delta\gamma \\ \delta\gamma^2 \\ \delta\gamma^3 \\ \delta\gamma^4 \\ \delta\gamma^5 \\ \delta\gamma^6 \end{bmatrix} \otimes w$$



# Chromatic emittance growth

## New nonlinear matrix formalism

- Explicit form of nonlinear matrix for 1 term in  $\delta\gamma$  in  $x, u_x$  only:

Linear transfer matrix      Coupling of perturbation in energy

$$\mathcal{M} = \begin{bmatrix} M & -\frac{1}{\gamma}M_D \\ 0 & M \end{bmatrix} = \begin{bmatrix} \boxed{1 - \frac{\alpha_{xx}^2 \Delta t^2}{\gamma} \quad \frac{\Delta t}{\gamma}} & \boxed{\frac{\alpha_{xx}^2 \Delta t^2}{\gamma^2} \quad -\frac{\Delta t}{\gamma^2}} \\ \boxed{-\alpha_{xx}^2 \Delta t \quad 1} & \boxed{0 \quad 0} \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Explicitly, for 1 term:

$$w_\delta = \begin{bmatrix} 1 \\ \delta\gamma \end{bmatrix} \otimes w = \begin{bmatrix} x \\ u_x \\ \delta\gamma x \\ \delta\gamma u_x \end{bmatrix}$$

Linear transfer matrix

- For 6 orders, in block matrix form:

$$\mathcal{M} = \begin{bmatrix} M & -\frac{1}{\gamma}M_D & \frac{1}{\gamma^2}M_D & -\frac{1}{\gamma^3}M_D & \frac{1}{\gamma^4}M_D & -\frac{1}{\gamma^5}M_D & \frac{1}{\gamma^6}M_D & \dots \\ 0 & M & -\frac{1}{\gamma}M_D & \frac{1}{\gamma^2}M_D & -\frac{1}{\gamma^3}M_D & \frac{1}{\gamma^4}M_D & -\frac{1}{\gamma^5}M_D & \dots \\ 0 & 0 & M & -\frac{1}{\gamma}M_D & \frac{1}{\gamma^2}M_D & -\frac{1}{\gamma^3}M_D & \frac{1}{\gamma^4}M_D & \dots \\ 0 & 0 & 0 & M & -\frac{1}{\gamma}M_D & \frac{1}{\gamma^2}M_D & -\frac{1}{\gamma^3}M_D & \dots \\ 0 & 0 & 0 & 0 & M & -\frac{1}{\gamma}M_D & \frac{1}{\gamma^2}M_D & \dots \\ 0 & 0 & 0 & 0 & 0 & M & -\frac{1}{\gamma}M_D & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & M & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

But how many orders in the expansion are required?

# Chromatic emittance growth

## New extended beam matrix

- Now we want to use the nonlinear matrix to calculate the transformation of the phase space ellipsoid similar to

$$\sigma = M\sigma M^T$$

- For a particle beam with energy  $\gamma + \delta\gamma$

$$\sigma = M(\gamma + \delta\gamma)\sigma_0(\gamma + \delta\gamma)[M(\gamma + \delta\gamma)]^T$$

- The expansion of M up to the mth order can be obtained from the nonlinear matrix through

$$M(\gamma + \delta\gamma) \simeq \mathcal{P}^T \mathcal{M} \mathcal{G}$$

$$\mathcal{P} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \vdots \end{bmatrix} \otimes \mathbb{I}_n$$

$$\mathcal{G} = \begin{bmatrix} 1 \\ \delta\gamma \\ \delta\gamma^2 \\ \delta\gamma^3 \\ \vdots \end{bmatrix} \otimes \mathbb{I}_n$$

# Chromatic emittance growth

## New extended beam matrix

- Defining

$$M_\delta \equiv M(\gamma + \delta\gamma) \equiv \mathcal{P}^T \mathcal{M} \mathcal{G}$$

- For a particle beam with energy  $\gamma + \delta\gamma$

$$\sigma(\delta\gamma) = M_\delta \sigma_0 [M_\delta]^T$$

- We can integrate over a particle distribution in  $\delta\gamma$  to find an average beam matrix:

$$\langle \sigma \rangle = \int_{-\infty}^{\infty} \rho(\delta\gamma) \sigma(\delta\gamma) d(\delta\gamma) = \int_{-\infty}^{\infty} \rho(\delta\gamma) M_\delta \sigma_0 [M_\delta]^T d(\delta\gamma)$$

$$\langle \sigma \rangle = \mathcal{P}^T \mathcal{M} \left[ \int_{-\infty}^{\infty} f(\delta\gamma) (\mathcal{G} \sigma_0 \mathcal{G}^T) d(\delta\gamma) \right] \mathcal{M}^T \mathcal{P}$$

# Chromatic emittance growth

## New extended beam matrix

- This allows us to define an extended beam matrix:  $\Sigma_{ij} = \begin{cases} \langle \delta\gamma^2 \rangle^{\frac{i+j}{2}} \sigma_0 (i+j-1)!! & i+j \text{ even} \\ 0 & i+j \text{ odd} \end{cases}$

- e.g. for  $m=3$  
$$\Sigma = \begin{bmatrix} \sigma_0 & 0 & \sigma_0 \langle \delta\gamma^2 \rangle & 0 \\ 0 & \sigma_0 \langle \delta\gamma^2 \rangle & 0 & 3\sigma_0 \langle \delta\gamma^2 \rangle^2 \\ \sigma_0 \langle \delta\gamma^2 \rangle & 0 & 3\sigma_0 \langle \delta\gamma^2 \rangle^2 & 0 \\ 0 & 3\sigma_0 \langle \delta\gamma^2 \rangle^2 & 0 & 15\sigma_0 \langle \delta\gamma^2 \rangle^3 \end{bmatrix}$$

Which transforms as

$$\Sigma = \mathcal{M} \Sigma_0 \mathcal{M}^T$$

- And is related to the beam emittance via

$$\epsilon_N = \sqrt{\det(\mathcal{P}^T \mathcal{M} \Sigma \mathcal{M}^T \mathcal{P})}$$

# Chromatic emittance growth

## Test of extended beam matrix

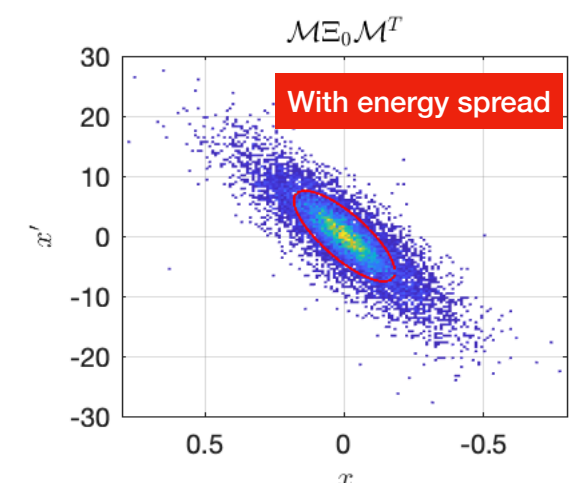
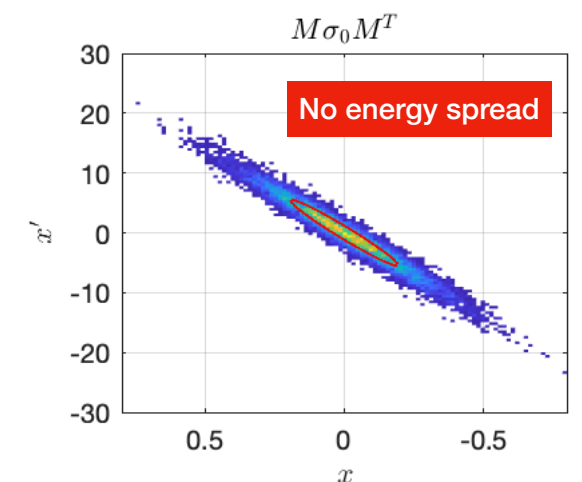
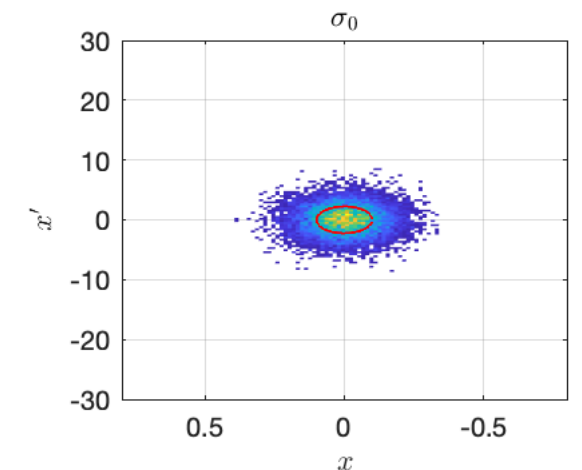
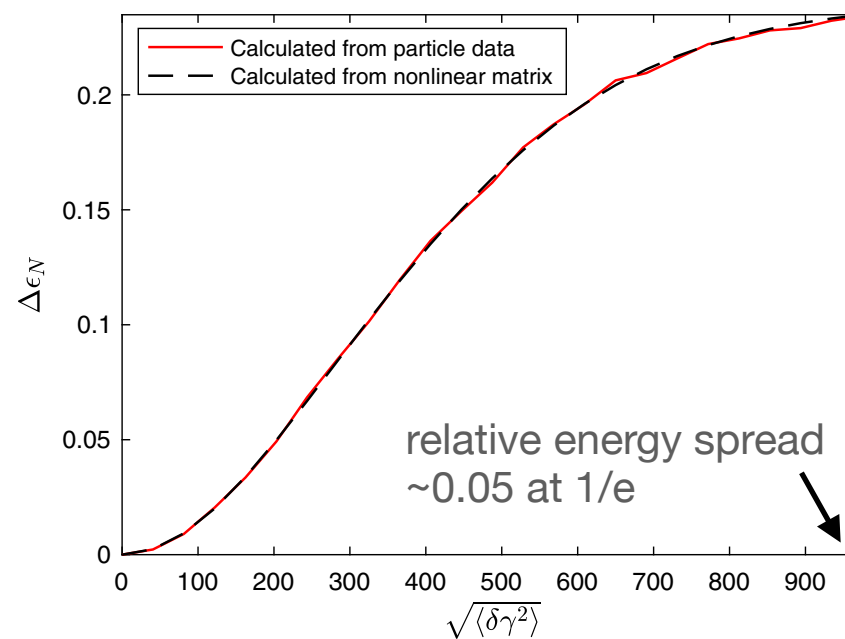
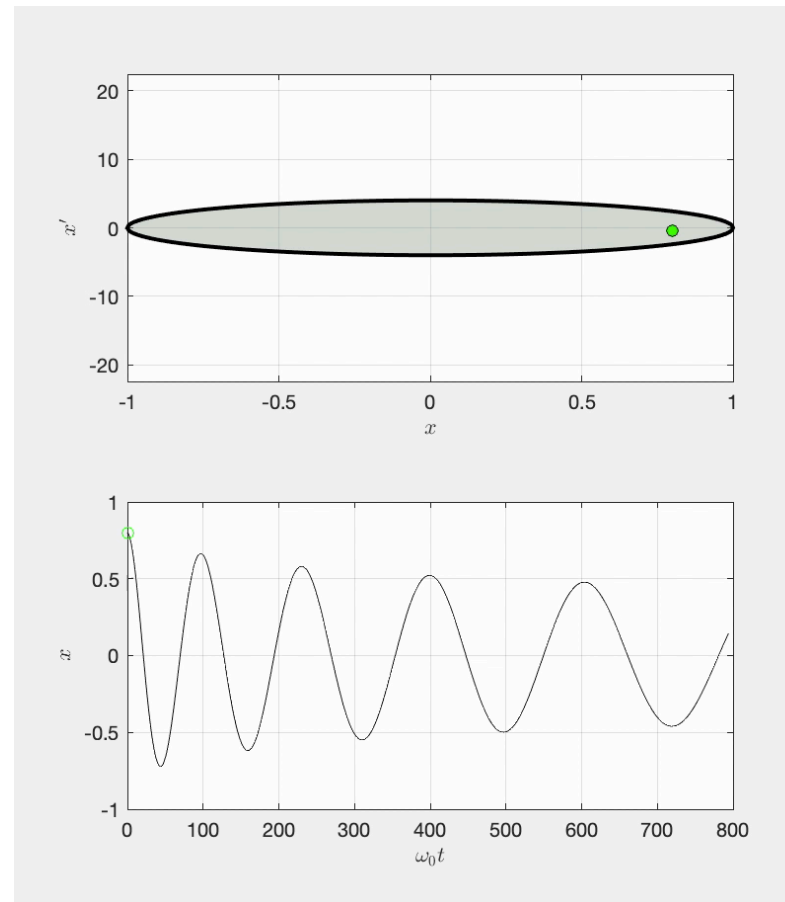
- Verification of method - results from nonlinear matrix / extended beam matrix compared with 100000 single particles

$$\epsilon_N = \sqrt{\langle x^2 \rangle \langle u^2 \rangle - \langle xu \rangle^2}$$

VS

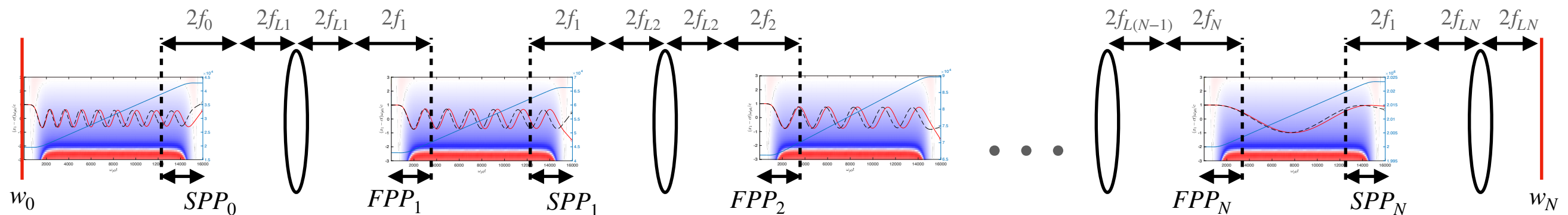
$$\epsilon_N = \sqrt{\det(\mathcal{P}^T \mathcal{M} \Sigma \mathcal{M}^T \mathcal{P})}$$

- Excellent agreement over large range of relative energy spreads



# A simple plasma accelerator lattice

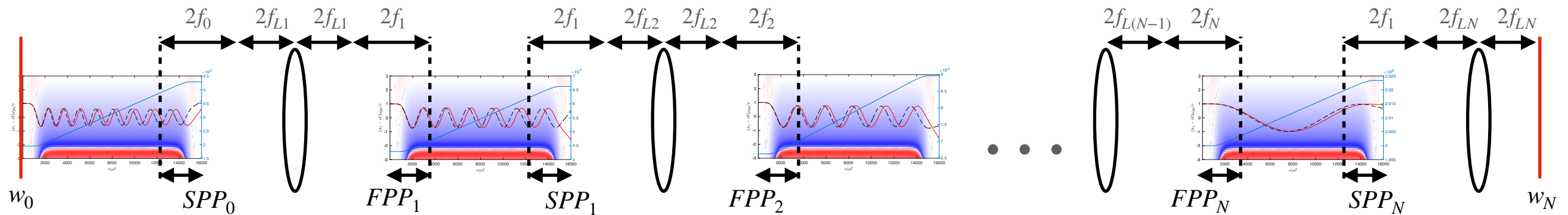
## Overview



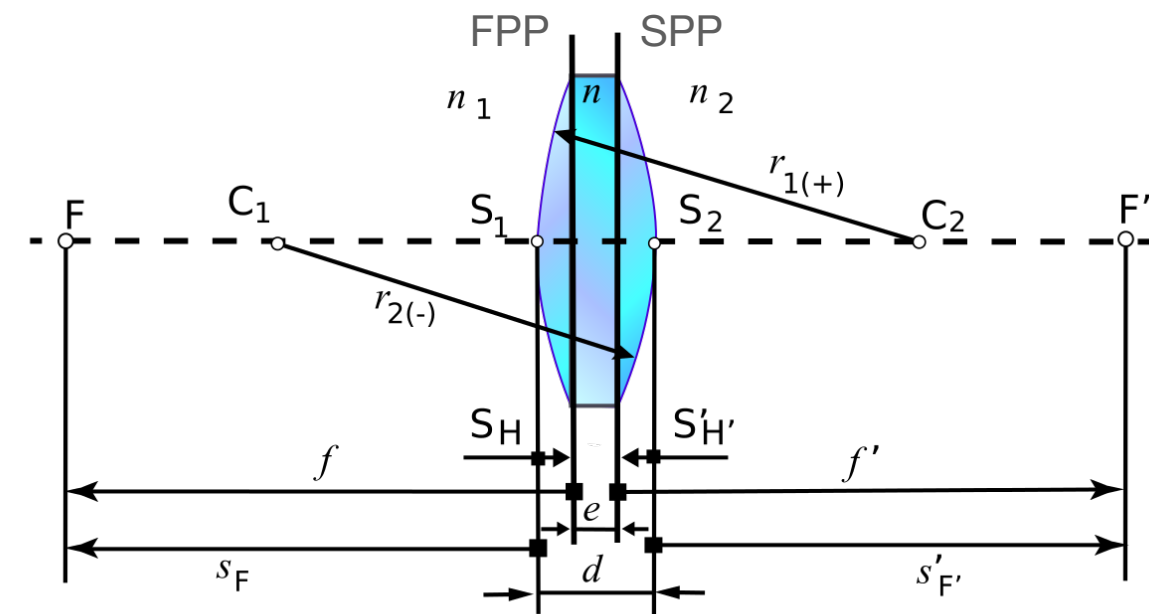
- **IF** involving staging, a plasma accelerator lattice may consist of
  - Plasma accelerating stages
  - Beam focusing optics between stages
  - Other elements (dechirpers etc.)
- We consider a simple  $N$ -element lattice with identical plasma stages and simple (but chromatic) lenses

# A simple plasma accelerator lattice

## Focusing properties



- Plasma accelerating stage effectively acts as a thick lens
- The first (FPP) and secondary (SPP) principal planes can be calculated from the matrix elements



Wikipedia

$$d_{\text{FPP}_s} = \frac{1 - \mathcal{M}_{2,2}^s}{\mathcal{M}_{2,1}^s} \gamma_{s-1} \quad d_{\text{SPP}_s} = \frac{1 - \mathcal{M}_{1,1}^s}{\mathcal{M}_{2,1}^s} \gamma_s$$

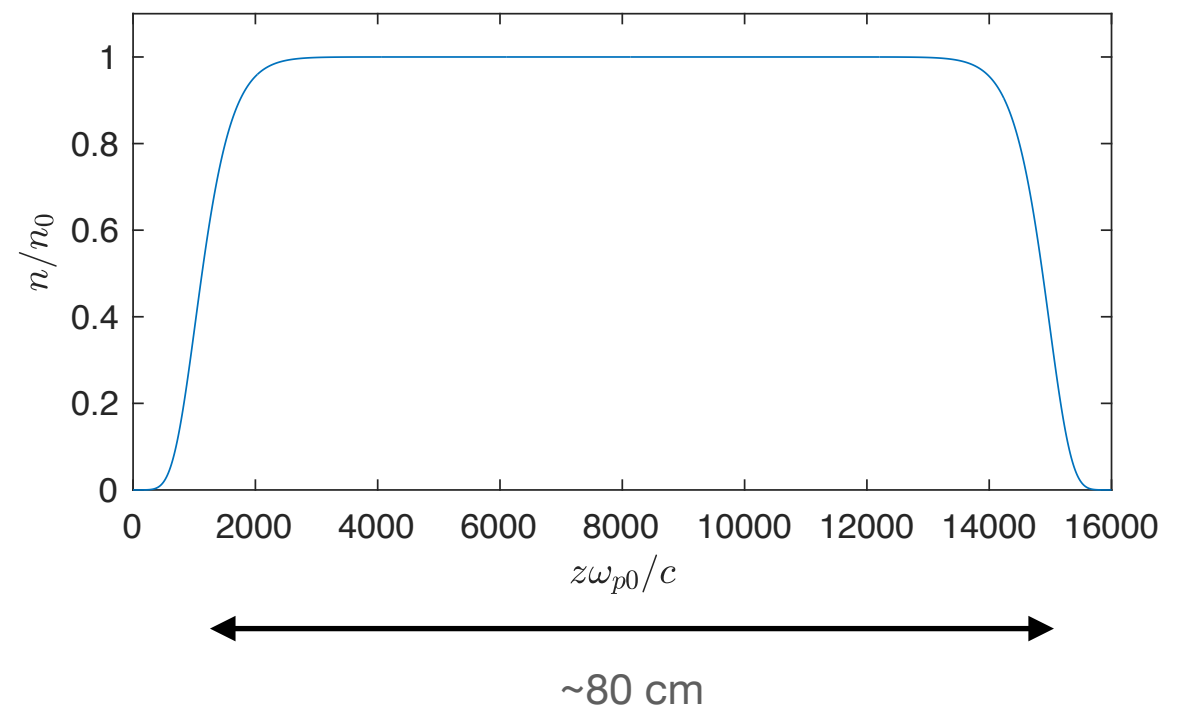
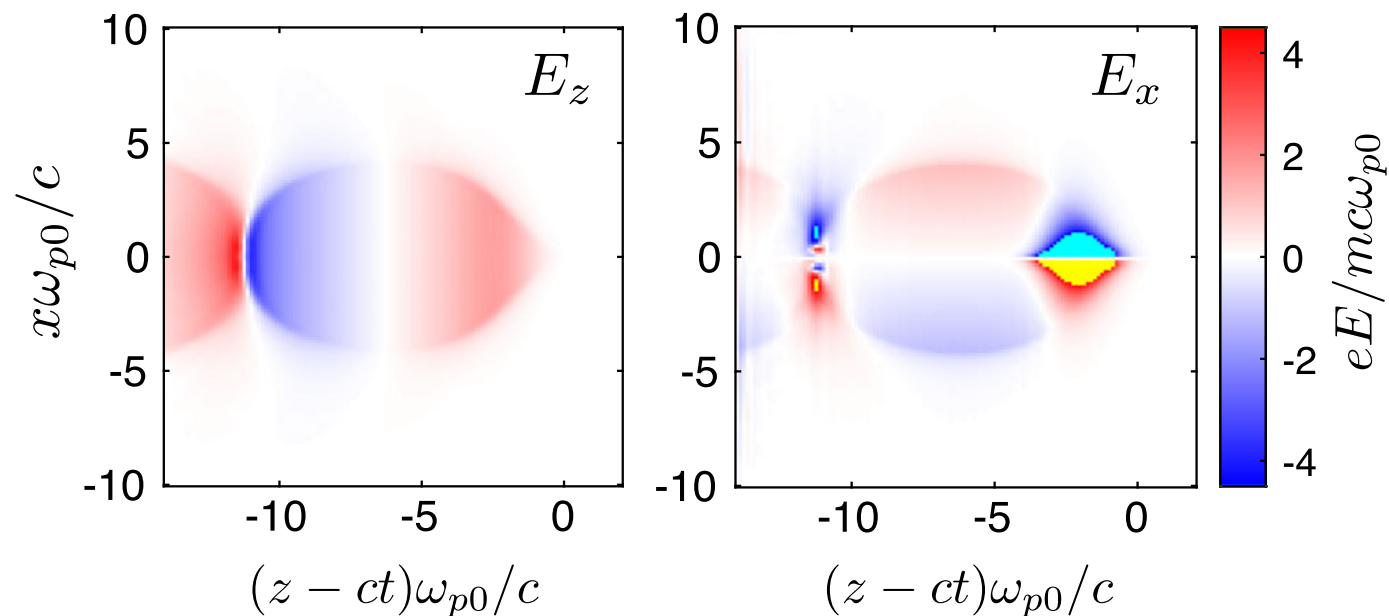
Advantage of the matrix approach -  
can easily find effective focal lengths  
of plasma accelerating stages



# A simple plasma accelerator lattice

## Plasma stage

- 3D simulation using the PIC code OSIRIS 4.0
- Beam driven (PWFA) chosen for simplicity
- Density profile with long ramps included for adiabatic matching<sup>1-4</sup> of beam into each accelerator stage



[1] K. Floettmann, Phys. Rev. ST Accel. Beams 17, 054402 (2014).

[2] I. Dornmair, K. Floettmann, and A. R. Maier, Phys. Rev. ST Accel. Beams 18, 041302 (2015).

[3] M. D. Litos, R. Ariniello, C. E. Doss, K. Hunt-Stone, and J. R. Cary, Phil. Trans. R. Soc. A 377, 20180181 (2019). [16] R. Ariniello, C. E. Doss, K. Hunt-Stone, J. R. Cary, and M. D. Litos, Phys. Rev. Accel. Beams 22, 041304 (2019).

[4] Y. Zhao, W. An, X. Xu, F. Li, L. Hildebrand, M. J. Hogan, V. Yakimenko, C. Joshi, and W. B. Mori, Phys. Rev. Accel. Beams 23, 011302 (2020).

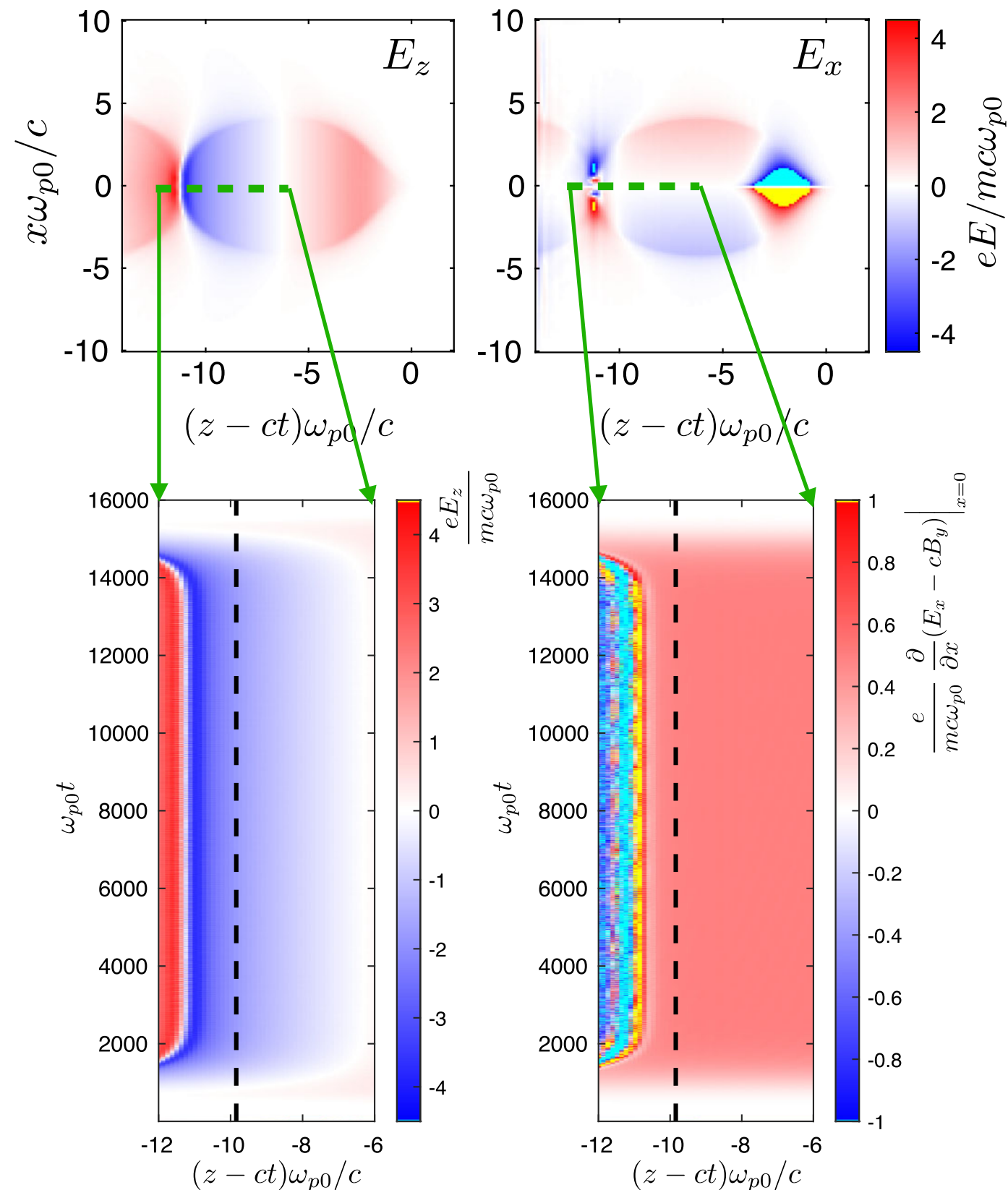
# A simple plasma accelerator lattice

## Plasma stage

- 3D simulation using the PIC code OSIRIS 4.0
- Each  $t^n = (0.6/\omega_{p0})n$  step the fields either side of the axis were dumped to get the field gradients:

$$[\alpha_{xx}^2]_j^n = -\frac{\partial}{\partial x} (E_x - cB_y) \Big|_{x=0,j}^n$$

$$\simeq -\frac{(E_x - cB_y)_{j,k+1,l}^n - (E_x - cB_y)_{j,k-1,l}^n}{2\Delta x}$$



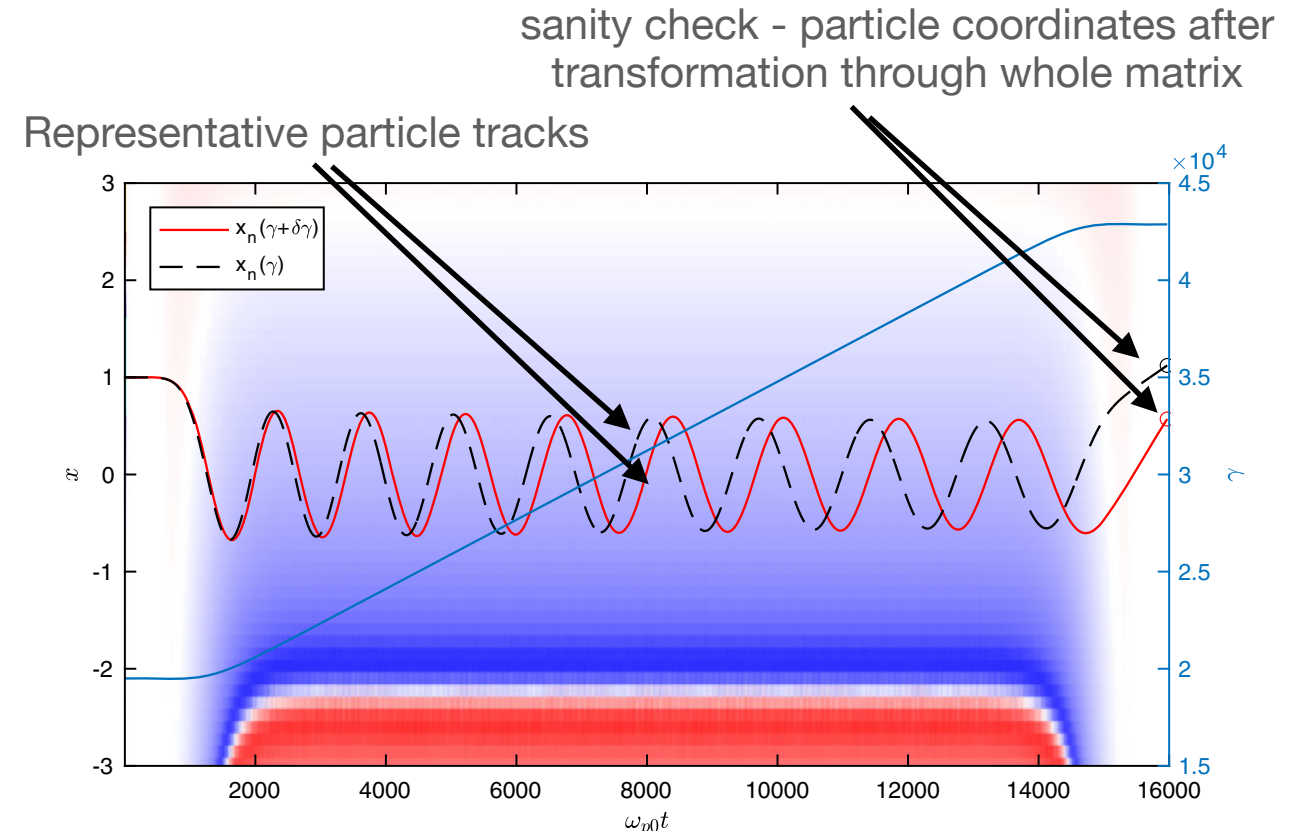
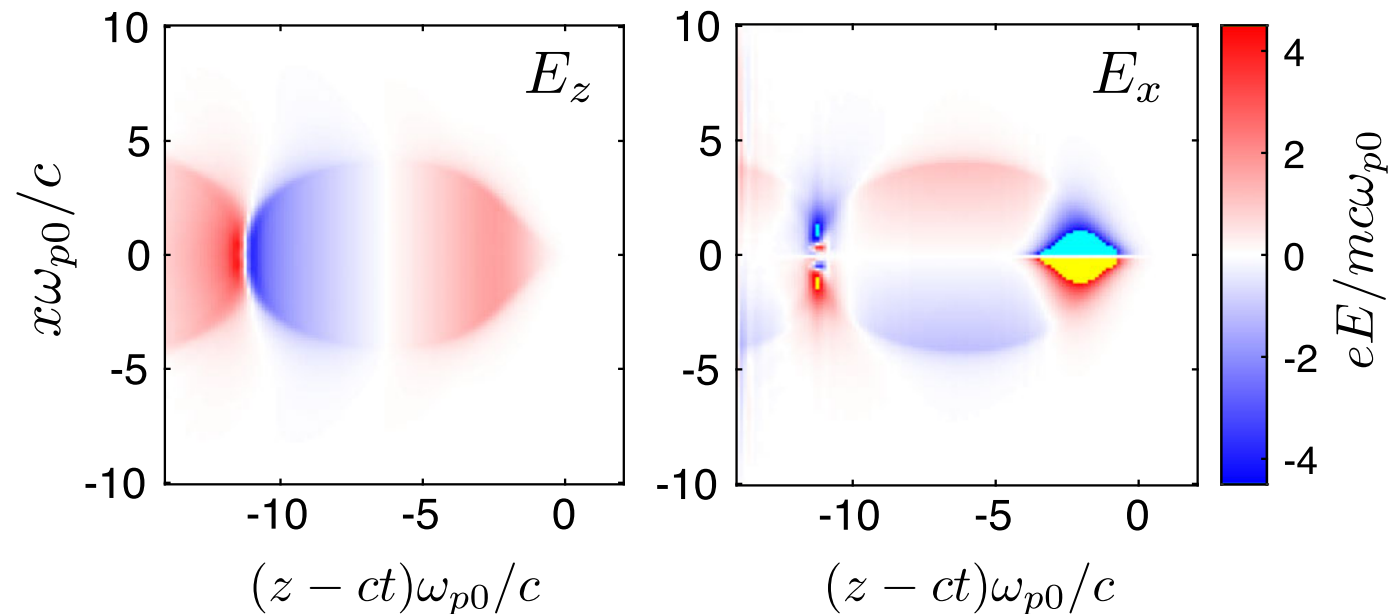
# A simple plasma accelerator lattice

## Plasma stage

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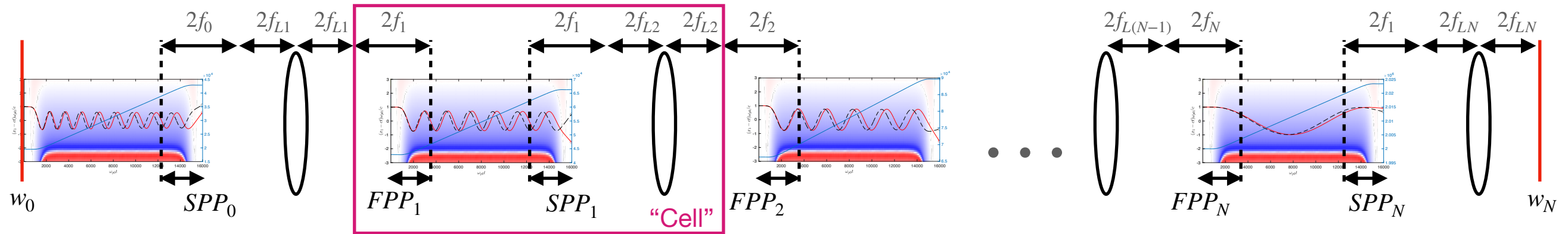
$$[\alpha_{xx}^2]_j^n = -\frac{\partial}{\partial x} (E_x - cB_y) \Big|_{x=0,j}^n$$

$$\simeq -\frac{(E_x - cB_y)_{j,k+1,l}^n - (E_x - cB_y)_{j,k-1,l}^n}{2\Delta x}$$



# A simple plasma accelerator lattice

## Focusing properties



- Each plasma stage “corrected” using negative drift space to act as simple thin lens\*

$$\mathcal{M}^{\star s} = \mathcal{D}_{\text{SPP}_s} \mathcal{M}^s \mathcal{D}_{\text{FPP}_s}$$

- Each “cell” of lattice consists of

1. Drift space for  $2f$  focusing of matrix
2. “Corrected” beam transport matrix
3. Drift space  $2f$  focusing of matrix
4. Drift space  $2f$  focusing of thin lens
5. (Chromatic) thin lens
6. Drift space  $2f$  focusing of thin lens

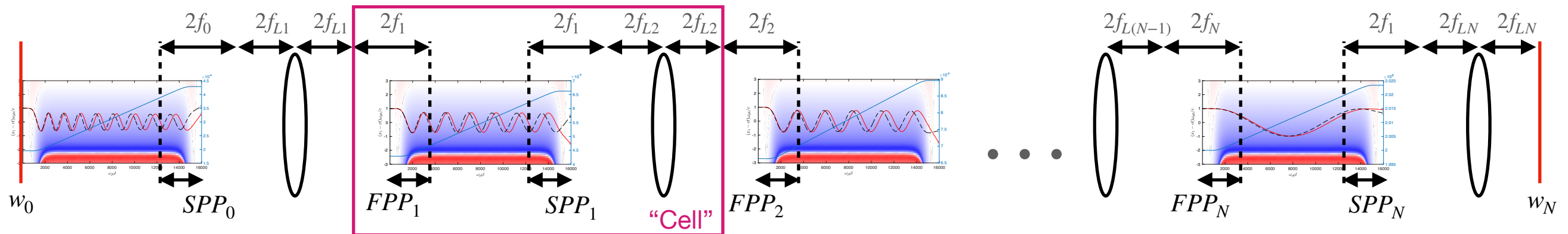
$$\mathcal{C}^s = \mathcal{D}_{2f_L}^s \mathcal{F}_L^s \mathcal{D}_{2f_L}^s \mathcal{D}_{2f_s} \mathcal{M}^{\star s} \mathcal{D}_{2f_{s-1}}$$

\* checked to ensure drift space between end of accelerator and lens is positive!

Note that this is just for designing the lattice, it doesn't mean that the calculation itself is simplified

# A simple plasma accelerator lattice

## Physical parameters



- Full lattice described by

$$\mathcal{C} = \prod_{s=0}^{N_{\text{stages}}} \mathcal{C}^s$$

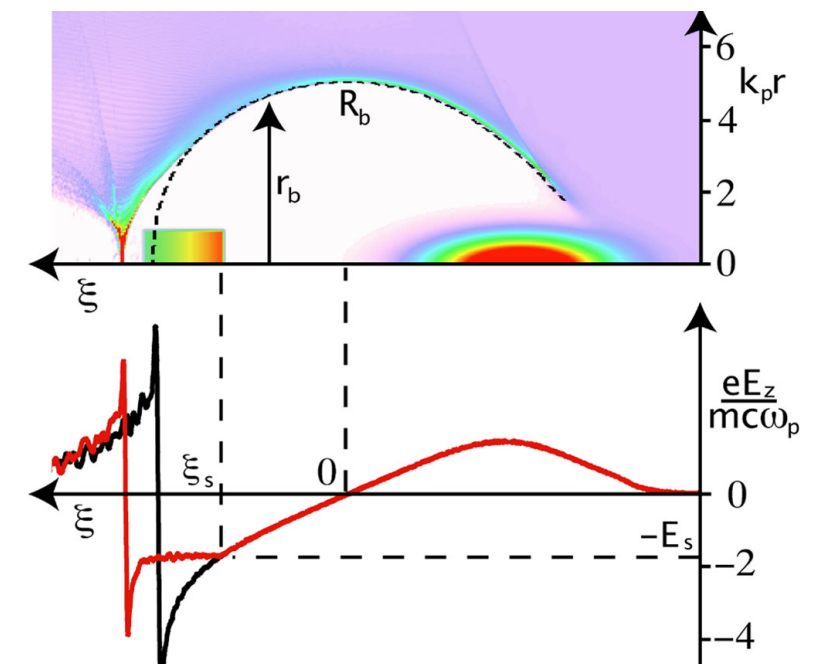
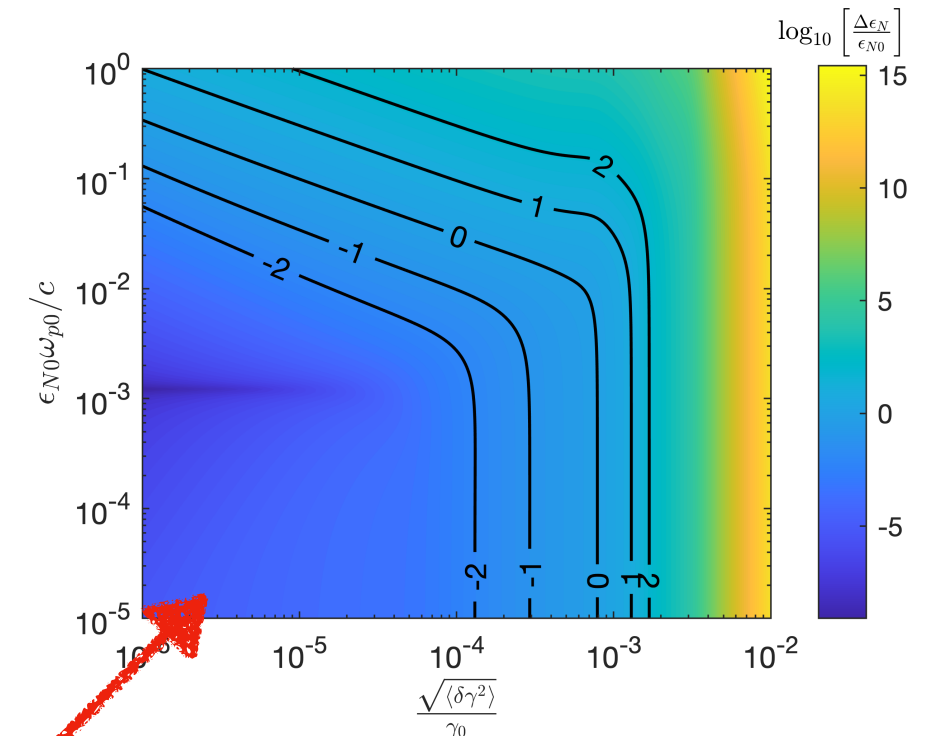
- Lengths normalized to  $c/\omega_{p0}$
- If  $10^{16} \text{ cm}^{-3}$  plasma chosen as reference density then plasma accelerator length 85 cm
- Lens focal length 42 cm initially, then scaling as  $\sqrt{\gamma_s/\gamma_0}$
- Total length of accelerator is 740 meters\*

(\* Average accelerating gradient of 1.8 GeV/m)

# Results of study

## Chromatic slice emittance growth

- Using the nonlinear matrix allows us to scan a large parameter space through the entire accelerating structure to 1 TeV
- For a perfectly beam loaded wake<sup>1</sup> in the blowout regime, it could represent dynamics of the whole bunch
- Study relative chromatic emittance growth as a function of relative energy spread and initial normalized emittance
- For initial relative energy spreads below  $10^{-3}$  and normalized emittance below mm-mrad, the chromatic emittance growth can be minimal (less than the initial emittance)

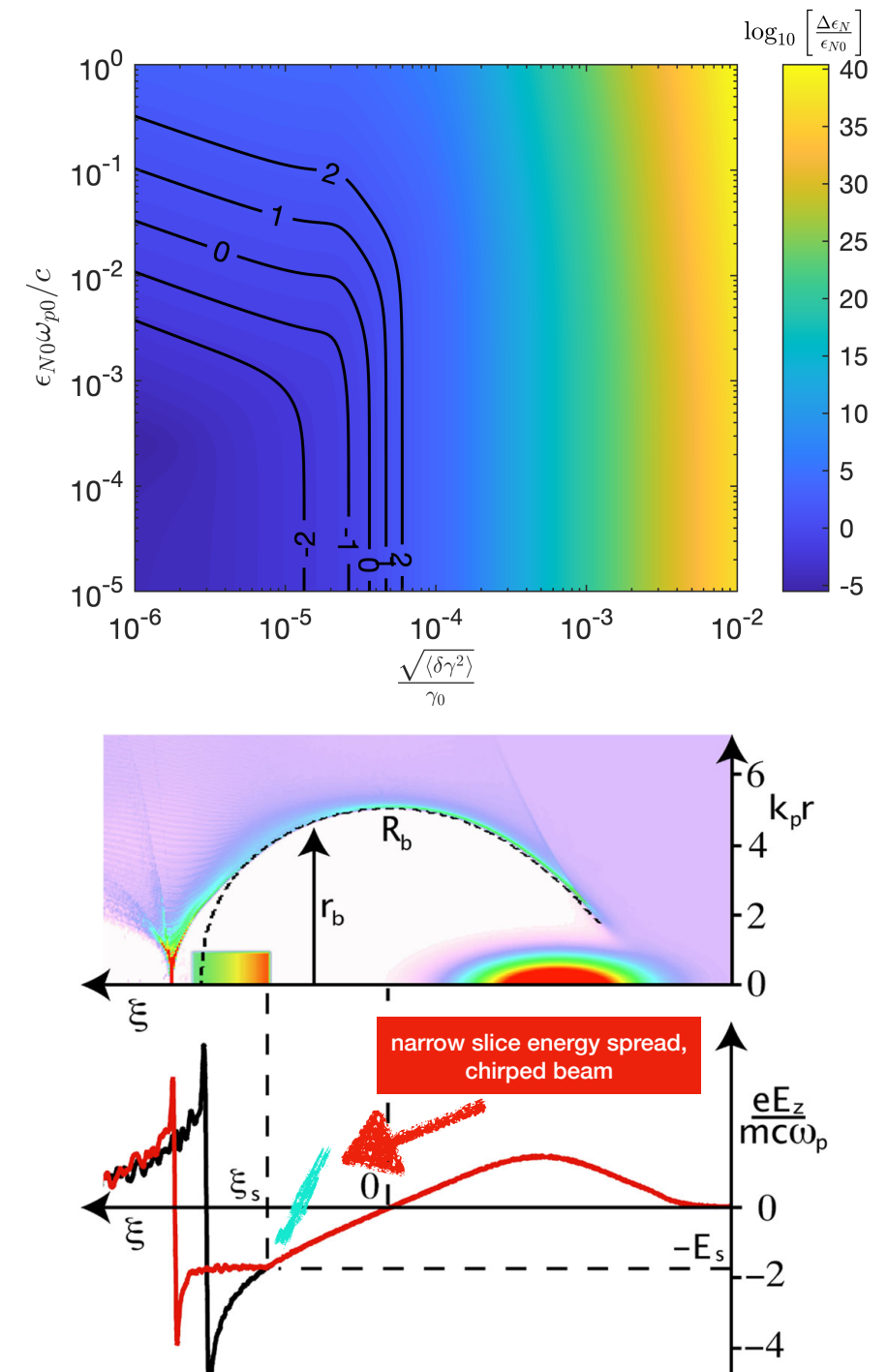




# Results of study

## Chromatic emittance growth of finite beam

- Study same parameter space for a **zero slice-emittance, chirped beam**
- Energy distribution for gaussian temporal profile and linear chirp is also gaussian
- Can use same theory for finite beam with linear chirp - but now constant relative energy spread (instead of absolute energy spread)
- For relative energy spread (energy spread growth per stage below  $10^{-5}$  and normalized emittance below mm-mrad, the chromatic emittance growth can be minimal (less than the initial emittance)



# Summary

## Conclusions

1. Standard beam dynamics techniques may be applied to data from particle-in-cell simulations in moving window
2. By extending the beam matrix / transfer matrix approach to include nonlinear terms in energy spread, we can characterize chromatic emittance growth
3. A study of a **simple** lattice consisting of identical, adiabatically matched beam driven plasma accelerating stages connected with beam optics places limits on energy spread and emittance to limit chromatic emittance growth:
  - Normalized emittance below mm-mrad
  - Absolute energy spread below  $10^{-3}$  of initial energy
  - Relative energy spread (energy spread growth per stage) **below  $10^{-5}$**



# Summary

## End notes

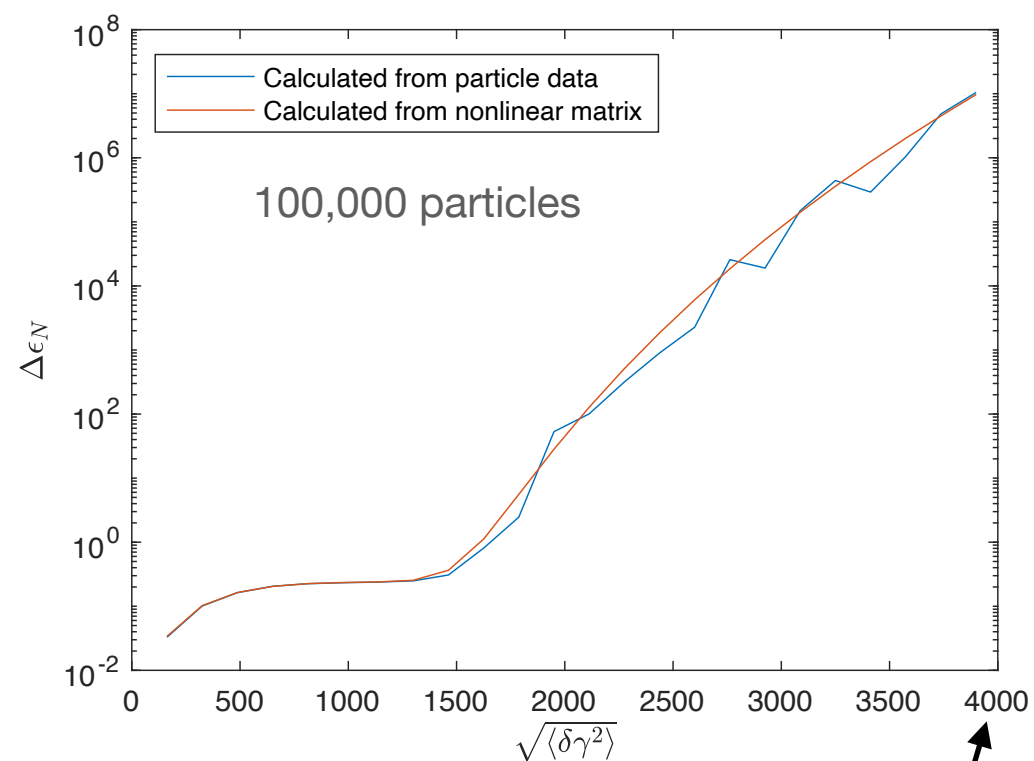
- **This study was not meant to be definitive!**
- any other considerations - e.g. schemes with reduced transverse fields (hollow channels) - this method can be used for these too!
- Aim was just to show that it was possible to design a plasma based accelerator using the methods of conventional accelerators all the way to 1 TeV
- Hope is that method is useful to spur future design work (I'm happy to share code)
  - understanding the role of lenses
  - energy spread control
  - extensions to nonlinear fields, curved plasma accelerators ....
  - application of AI methods - automatically differentiated framework with Bayesian optimization....

... and beyond!

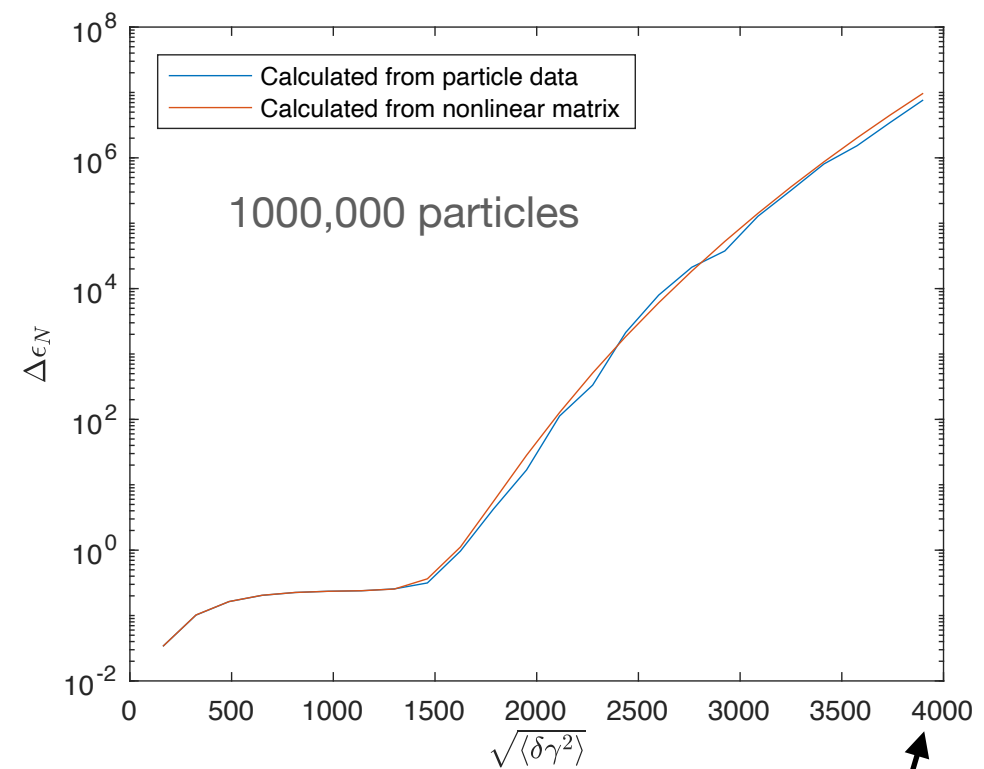


# EXTRAS

- Method accurate over large range - comparison limited by particle statistics



relative energy spread 34%



relative energy spread 34%

# Transfer matrix

- For  $M$  transfer matrix, MD  $dM/d\gamma$ :

$$\mathcal{M} = \mathbb{I}_{m+1} \otimes M + \Gamma \otimes M_D$$

$$\Gamma_{a,b} = \begin{cases} \left(-\frac{1}{\gamma}\right)^{b-a} & b > a \\ 0 & \text{otherwise} \end{cases}$$

# Drift space, focusing

- Drift space - has gamma dependence

$$d = \exp(A_1 L / c) = \mathbb{I}_n + A_1 \frac{L}{c}$$

- Extended drift space

$$\mathcal{D} = \mathbb{I}_{m+1} \otimes d + \Gamma \otimes A_1 \frac{L}{c}$$

- Thin lens - no gamma dependence in  $x$  or  $u_x$  coordinates

$$F = \mathbb{I}_{n/2} \otimes \begin{bmatrix} 1 & 0 \\ -\frac{c}{f^*} & 1 \end{bmatrix}$$

- Extended thin lens

$$\mathcal{F} = \mathbb{I}_{m+1} \otimes F$$

# Chromatic emittance growth

## Accuracy of the nonlinear matrix

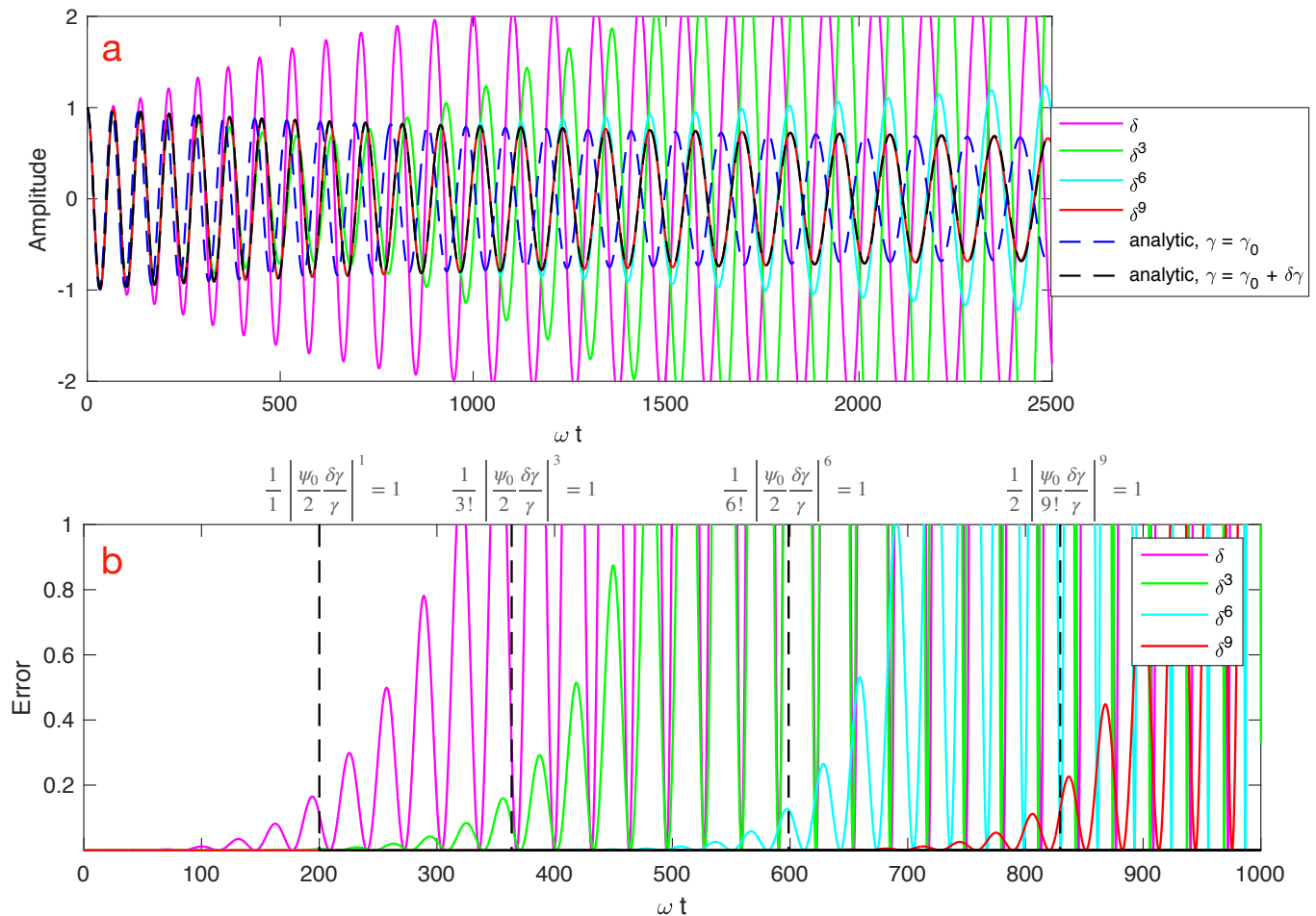
- But how many orders in the expansion are required?

- For a particle with energy  $\gamma + \delta\gamma$ , eigenmodes for betatron oscillations given by

$$\exp(\pm i \int_0^t \alpha_{xx}(t') dt' / \sqrt{\gamma(t') + \delta\gamma})$$

- Can expand in  $\delta\gamma$

$$\exp \left[ \pm i \psi_0 \pm i \int_0^{\psi_0} d\psi \left( -\frac{1}{2} \frac{\delta\gamma}{\gamma} + \frac{3}{4} \frac{\delta\gamma^2}{\gamma^2} + \dots \right) \right]$$



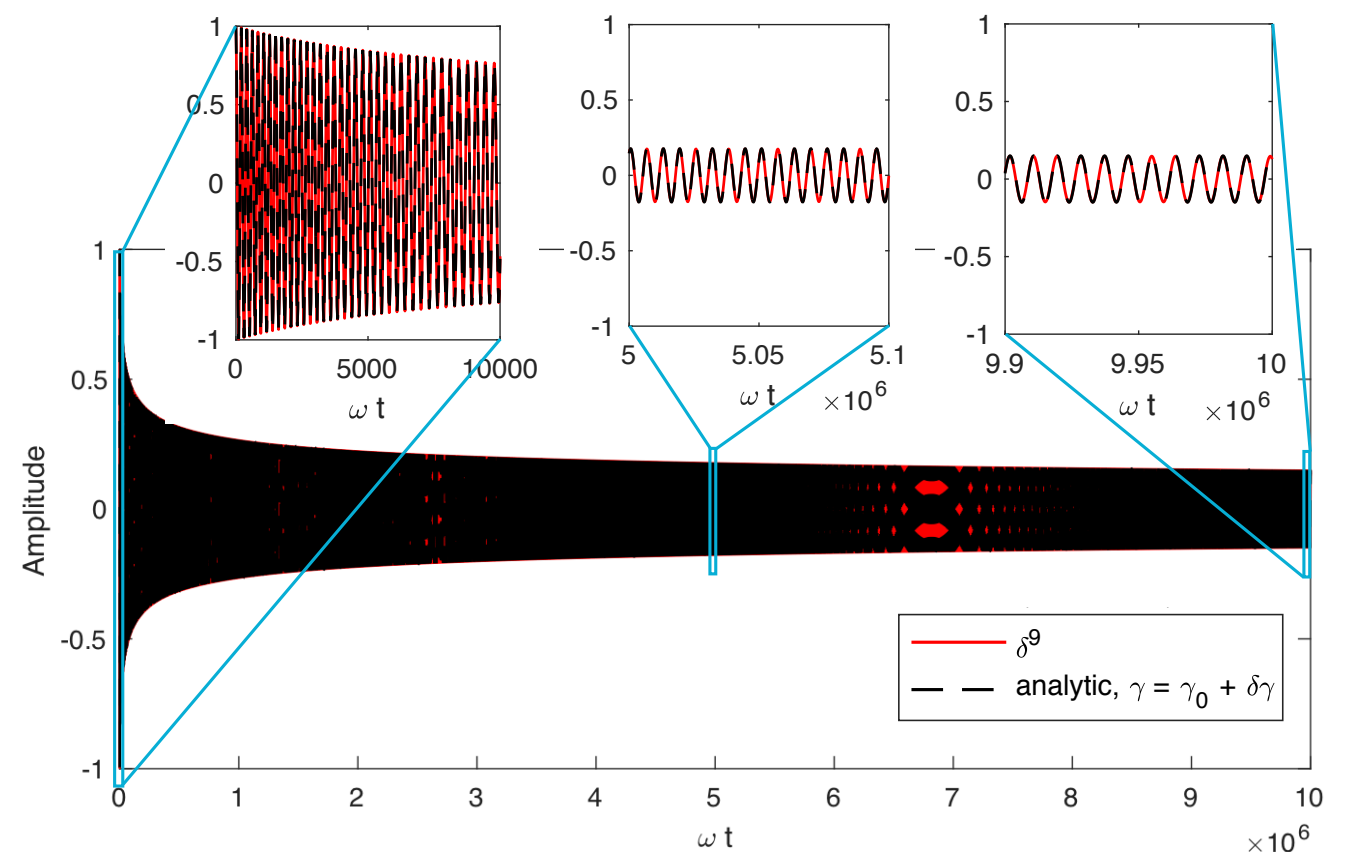
- For highest order in expansion  $m$ :

$$\left| \frac{\delta\gamma}{2} \int_0^{\psi_0} \frac{d\psi}{\gamma} \right|^m \ll m!$$

# Chromatic emittance growth

## Accuracy of the nonlinear matrix

- Demonstration of method accuracy; Betatron oscillations for particle linearly accelerated with design energy from 0.5 GeV to 1 TeV compared to WKB solution
- Time in units of  $1/\omega$ , where betatron frequency is  $\omega/\sqrt{\gamma}$
- Deviation from the design energy of  $\delta\gamma/\gamma = 2\%$
- Including terms up to  $m=9$



(~500 meters if density  $\sim 10^{16} \text{ cm}^{-3}$ )