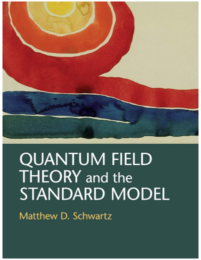
The Landau Bootstrap:
from Singularities to Scattering Amplitudes

Andrew McLeod

SLAC Theory Seminar
January 7, 2026

Scattering Amplitudes

Scattering amplitudes are objects of high experimental and theoretical interest in QFT



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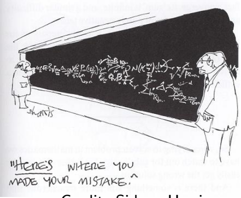

Credit: Sidney Harris

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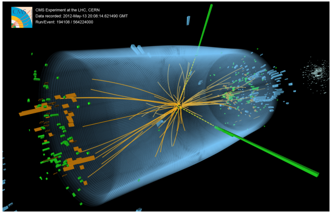


- encode the probability of different configurations of particles scattering into each other, allowing us to make precision predictions for **collider experiments**

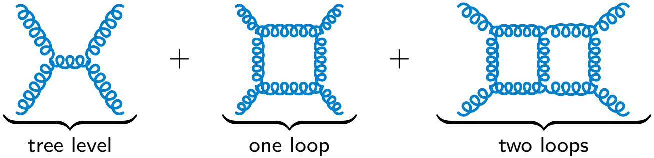
$$\frac{d\sigma}{d\Omega} = \int |\mathcal{A}|^2$$

- enter predictions about the **large-scale structure of the universe** and **gravitational waves**

Feynman Diagrams

Feynman diagrams provide an intuitive picture for calculating amplitudes perturbatively

$$\mathcal{A}_{2\to 2} =$$



$$+ \cdots$$

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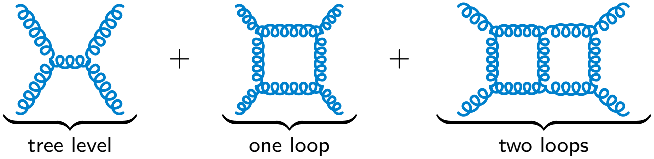


$+ \cdots$

Two sources of difficulty arise when using Feynman diagrams

- number of diagrams grows exponentially

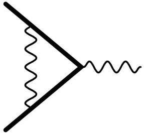
$gg \to\ ??$	gg	ggg	$gggg$	$ggggg$	$gggggg$
# tree diagrams	4	25	220	2485	34300

[Mangano, Parke (1990)]

- at loop level, each diagram becomes a complicated integral over loop momenta

Feynman Diagrams

In fact, these integrals are so difficult that while one might encounter a Feynman diagram in a first-year quantum field theory course that looks like this...



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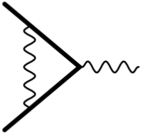
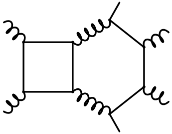
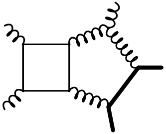


...the state of the art looks like this:

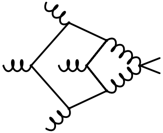
state of the art



[Henn, Peraro, Xu, Zhang (2022)]



[Badger, Becchetti, Chaubey, Marzucca (2023)]

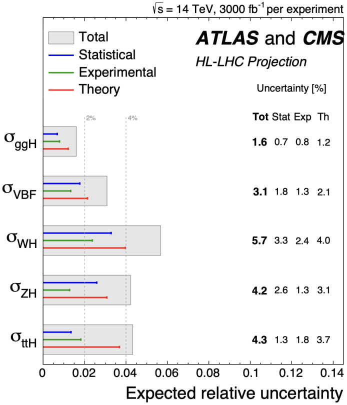


[Abreu, Chicherin, Ita, Page, Sotnikov, Tschernow, Zoia (2023)]

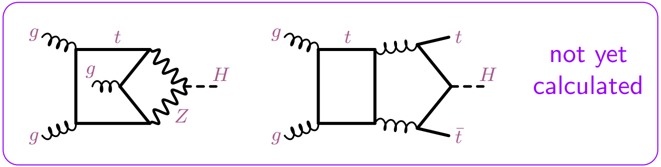
- As a general rule, we are only capable of computing diagrams where the number of loops L, external particles n_e, and masses n_m satisfy $L + n_e + n_m \lesssim 8$

State of the Art

As a result, Standard Model predictions for many processes are **only known with $\mathcal{O}(10\%)$ precision**—**not the $\mathcal{O}(1\%)$ precision that experimentalist aim to achieve** at the LHC



[Cepeda et al. (2019)]



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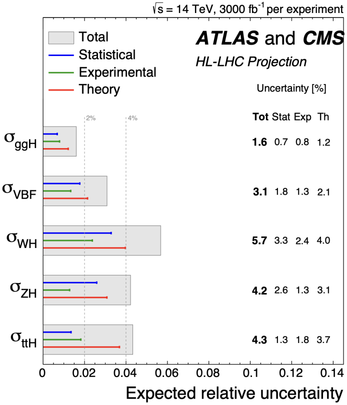

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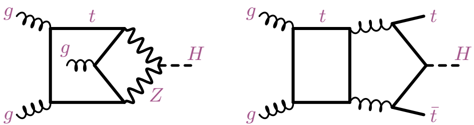

not yet calculated

The **evaluation of amplitudes remains one of the bottlenecks** to **measuring Standard Model parameters** with higher precision, and thus to **searching for new physics at the LHC**

Surprising Simplicity of Scattering Amplitudes

Despite this computational complexity, scattering amplitudes can be **strikingly simple**

- At tree level, the n-particle maximum-helicity-violating (MHV) gluon amplitude is simply

$$\left|\mathcal{A}_n(p_1^-, p_2^-, p_3^+, \ldots, p_n^+)\right|^2 \propto \sum_{\sigma \in S_n} \frac{(p_1 \cdot p_2)^4}{(p_{\sigma_1} \cdot p_{\sigma_2})(p_{\sigma_2} \cdot p_{\sigma_3}) \cdots (p_{\sigma_n} \cdot p_{\sigma_1})}$$

[Parke, Taylor (1986)]

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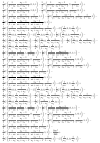
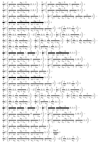
- Similar simplifications occur at loop level

The **two-loop six-particle MHV amplitude** in planar $\mathcal{N} = 4$ supersymmetric Yang-Mills theory



$+ \cdots +$



$$\Rightarrow \quad R_6^{(2)}(u_1, u_2, u_3) = \sum_{i=1}^{3} \left(L_4(x_i^+, x_i^-) - \frac{1}{2} \mathrm{Li}_4(1 - 1/u_i) \right) - \frac{1}{8} \left(\sum_{i=1}^{3} \mathrm{Li}_2(1 - 1/u_i) \right)^2 + \frac{1}{24} J^4 + \frac{\pi^2}{12} J^2 + \frac{\pi^4}{72}. \quad (3)$$

was first computed as a 17 page expression and later simplified to a two-line expression

[Del Duca, Duhr, Smirnov (2009)] [Goncharov, Spradlin, Vergu, Volovich (2010)]

Surprising Simplicity of Scattering Amplitudes

This result spotlighted a surprising fact: at two loops, the six-particle amplitude can be expressed in terms of **multiple polylogarithms** with a **very restricted set of singularities**

- **Multiple polylogarithms** are iterated integrals whose derivatives are logarithmic:

$$dF = \sum_i F^{s_i} d\log s_i$$

where the s_i are algebraic functions, and each F^{s_i} is also a **multiple polylogarithm**

Examples of such functions include $\log(z)$ and $\mathrm{Li}_m(z)$:

$$\mathrm{Li}_1(z) = -\log(1-z), \quad \mathrm{Li}_m(z) = \int_0^z \frac{\mathrm{Li}_{m-1}(t)}{t} dt$$

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- Moreover, the amplitude has **uniform transcendental weight four**, meaning each term involves precisely four logarithmic integrations (such as $\log^4(z)$, $\mathrm{Li}_2(y)\mathrm{Li}_2(z)$, $\mathrm{Li}_4(z)$, ...)

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- The specific **logarithmic arguments** that appear in the amplitude turn out to be drawn from a very restricted set:

$$s_i \in \{u, v, w, 1-u, 1-v, 1-w, y_u, y_v, y_w\}$$

where, in terms of the Mandelstam variables $s_{i\ldots k} = (p_i + \cdots + p_k)^2$,

$$u = \frac{s_{12}s_{45}}{s_{123}s_{345}}, \qquad v = \frac{s_{23}s_{56}}{s_{234}s_{123}}, \qquad w = \frac{s_{34}s_{61}}{s_{345}s_{234}}$$

$$y_u = \frac{1+u-v-w-\sqrt{(1-u-v-w)^2-4uvw}}{1+u-v-w+\sqrt{(1-u-v-w)^2-4uvw}}, \quad y_v = y_u\big|_{u\to v\to w\to u}, \quad y_w = y_u\big|_{u\to w\to v\to u}$$

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Taken together, these properties identify a relatively small space of functions

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Key Idea $\Rightarrow$ if we can identify (or construct) the space of functions that we expect an amplitude to evaluate to, we can try to **bootstrap** it within that space

basis of functions

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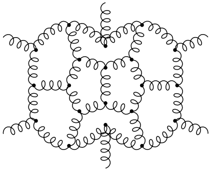
- Starting from a general ansatz, impose appropriate **symmetries and factorization properties** (collinear, soft, Regge, multi-particle factorization limits, . . .)
- Determine the value of all remaining coefficients by matching to **expansions of the amplitude around singular limits** (threshold expansions, large/small mass limits, . . .)

Perturbative Bootstrap Methods

This strategy has proven wildly successful in planar $\mathcal{N} = 4$ supersymmetric Yang-Mills theory

planar $\mathcal{N} = 4$ supersymmetric Yang-Mills



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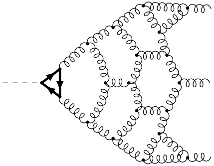

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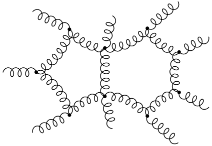

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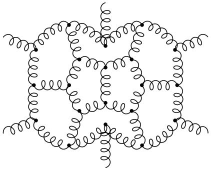

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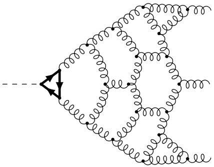

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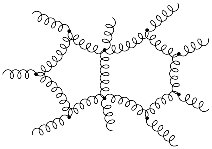

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...with one important caveat:

In planar $\mathcal{N} = 4$ we got unreasonably lucky, and **could guess** the correct spaces of functions

$\Rightarrow$ for instance, in the six-particle amplitude one encounters **the same class of special functions**, but **with higher transcendental weight** at higher loop orders

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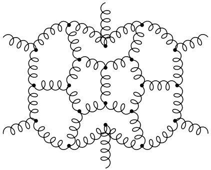

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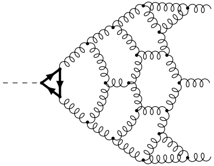

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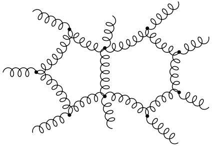

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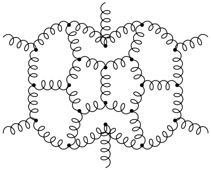

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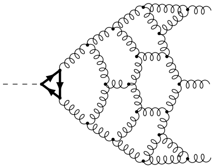

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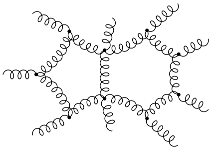

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- Extending these methods to more realistic theories requires being able to **predict** what types of functions will appear (and developing the technology for **working** with these functions)
- While predicting these function spaces for full amplitudes remains difficult, we have gotten increasingly good at characterizing these spaces for **individual Feynman integrals**

The Landau Bootstrap

In fact, I will argue that **our increasingly-rich understanding of the singularities and discontinuities** of Feynman integrals is, in many cases, sufficient to **bootstrap them directly**

[Hannesdottir, **AJM**, Schwartz, Vergu (2024)]

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[Hannesdottir, **AJM**, Schwartz, Vergu (2024)]

(i) Identify **where the integral can become singular** using the Landau equations, and investigate how the integral behaves near each singularity

(ii) **Characterize the types of functions** an integral evaluates to by computing its leading singularities

(iii) Place **constraints on where and how each singularity can appear** in the final answer

(iv) Use information about the **expansion of the integral around singular points** in order to determine the value of all remaining free coefficients

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I will discuss each of these steps in more detail, focusing on recent developments in (iii)

The Landau Equations

Given an L-loop Feynman integral associated with the diagram G,

$$\mathcal{I}_G(p) = \int_{-\infty}^{\infty} d^D\ell_1 \cdots d^D\ell_L \frac{\mathcal{N}(\ell_i, p)}{\prod_{e\in G}(q_e(\ell_i,p)^2 - m_e^2 + i\epsilon)}$$

our first task is to determine where singularities and branch cuts can arise

$$\left\{ \begin{array}{r l} p & \text{– collection of external momenta} \\ q_e & \text{– momentum flowing through edge } e \\ m_e & \text{– mass of edge } e \end{array} \right\}$$

- These singular locations are described by solutions to the **Landau Equations** [Landau (1959)] [Nakanishi (1959)] [Bjorken (1959)]

$$\alpha_e(q_e^2 - m_e^2) = 0 \qquad \text{for every edge } e \text{ in } G$$

$$\sum_{e\in\text{loop}} \alpha_e q_e^\mu = 0 \qquad \text{for each independent loop in } G$$

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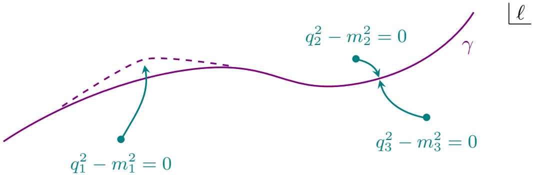

- More specifically, the Landau equations identify configurations in which the zeros in the denominator can pinch the integration contour γ
- Efficient packages now exist for identifying where solutions to these equations can arise
  [Panzer (2014)] [Correia, Giroux, Mizera (2025)]

Multiple Polylogarithms

When we only encounter integrals over logarithmic integration kernels, the answer can be expressed in terms of **multiple polylogarithms**

- We recall these functions have the property that their derivatives are logarithmic:

$$dF = \sum_i F^{s_i} d\log s_i$$

  where the s_i are algebraic functions, and the functions F^{s_i} are also multiple polylogarithms

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For instance, recalling the definition of $\mathrm{Li}_m(z)$ we can read off its symbol:

$$\mathrm{Li}_1(z) = -\log(1-z), \quad \mathrm{Li}_m(z) = \int_0^z \frac{\mathrm{Li}_{m-1}(t)}{t} dt$$

$$\mathcal{S}\big(\mathrm{Li}_m(z)\big) = -(1-z) \otimes z \otimes \cdots \otimes z$$

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- Importantly, if we instead read the symbol from left to right, we learn about the sequences of logarithmic discontinuities that each function encodes:

$$\mathcal{S}(F) = \sum_i s_i \otimes \mathcal{S}(F_{s_i}) \qquad \Rightarrow \qquad \text{Disc}_{s_j}(F) = (2i\pi)F_{s_j} + \mathcal{O}\big((2i\pi)^2\big)$$

Multiple Polylogarithms

When we only encounter integrals over logarithmic integration kernels, the answer can be expressed in terms of **multiple polylogarithms**

- We recall these functions have the property that their derivatives are logarithmic:

$$dF = \sum_i F^{s_i} d\log s_i$$

where the s_i are algebraic functions, and the functions F^{s_i} are also multiple polylogarithms

- In practice we often work in terms of the **symbol** of polylogarithms:

$$\mathcal{S}(F) = \sum_i \mathcal{S}(F^{s_i}) \otimes s_i$$

- Importantly, if we instead read the symbol from left to right, we learn about the sequences of logarithmic discontinuities that each function encodes:

$$\mathcal{S}(F) = \sum_i s_i \otimes \mathcal{S}(F_{s_i}) \qquad \Rightarrow \qquad \text{Disc}_{s_j}(F) = (2i\pi) F_{s_j} + \mathcal{O}\big((2i\pi)^2 \big)$$

note that Disc_{s_j} exposes the discontinuities of F_{s_j}, which live on the second Riemann sheet

Constructing an Ansatz

To construct an ansatz for a polylogarithmic Feynman integral, then, we thus proceed as follows:

- Identify where the Feynman integral can become singular, and characterize the behavior of the integral near these singularities
- **Construct the set of symbol letters** $\{s_i\}$ that are consistent with this singular behavior, using for instance the algorithm described in [Heller, von Manteuffel, Schabinger (2019)]
- **Build the space of multiple polylogarithms** that can be assembled using these symbol letters, to formulate an initial ansatz:*

$$\mathcal{S}(\mathcal{I}_G) = \sum c_{i_1,\dots,i_n} \, (s_{i_1} \otimes \cdots \otimes s_{i_n})$$

*Note this strategy can also be used when more complicated functions arise [Morales, Spiering, Wilhelm, Yang, Zhang (2022)]

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Importantly, each term in this ansatz encodes some sequence of discontinuities, **only some of which will be consistent with locality and causality**

*Note this strategy can also be used when more complicated functions arise [Morales, Spiering, Wilhelm, Yang, Zhang (2022)]

Constraints on Discontinuities

The discontinuities of Feynman integrals respect a number of different types of constraints:

α-positivity	which singularities appear in physically-realizable kinematics?	$\bullet \otimes \cdots$
adjacency constraints	which singularities are kinematically compatible with each other?	$\cdots \otimes \bullet \otimes \bullet \otimes \bullet \otimes \cdots$
hierarchical principle	which singularities arise from the same on-shell physics?	$\cdots \otimes \bullet \otimes \cdots \otimes \bullet \otimes \cdots$

- While these ideas have been around since the sixties, it is only in recent years that we have developed practical control over the constraints they place on specific examples

α-positivity

α-**positivity**	which singularities appear in physically-realizable kinematics?	● $\otimes \cdots$

These constraints follow from considering which singularities can appear on the principal Riemann sheet, in the Feynman parameter representation:

$$\mathcal{I}_G = \int_0^\infty \frac{d\alpha_1 \cdots d\alpha_E}{GL(1)} \int_{-\infty}^\infty d^D\ell_1 \cdots d^D\ell_L \frac{1}{\left(\sum_{e\in G} \alpha_e(q_e^2 - m_e^2)\right)^E}$$

- One can derive the same Landau equations from this representation:

$$\alpha_e(q_e^2 - m_e^2) = 0 \qquad \text{for every edge } e \text{ in } G$$

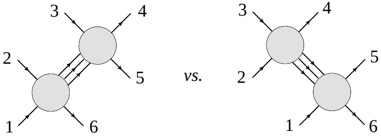
$$\sum_{e\in\text{loop}} \alpha_e q_e^\mu = 0 \qquad \text{for each independent loop in } G$$

- Since the original integration contour is over $\alpha_e \geq 0$, only the solutions that occur for non-negative, real values of α_e will be encountered by the original Feynman integral

Adjacency Constraints

adjacency constraints	which singularities are kinematically compatible with each other?	$\cdots \otimes \bullet \otimes \bullet \otimes \bullet \otimes \cdots$

- Causality implies the **Steinmann relations**, which tell us that Feynman integrals cannot have double discontinuities in partially overlapping channels [Steinmann (1960)] [Cahill, Stapp (1975)]



$$\text{Disc}_{s_{234}}(\text{Disc}_{s_{345}}(\mathcal{I})) = 0$$

$$\Updownarrow$$

~~$s_{345} \otimes s_{234}$~~ $\otimes \cdots$

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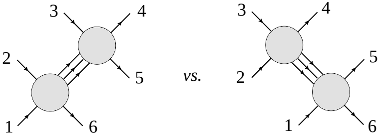

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$$\Updownarrow$$

$$~~s_{345} \otimes s_{234}~~ \otimes \cdots$$

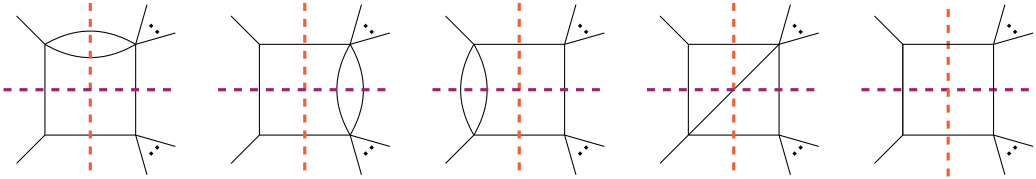
- Empirically, an **extended** version of these constraints have been seen to hold in massless planar amplitudes and Feynman integrals [Caron-Huot, Dixon, Dulat, von Hippel, **AJM**, Papathanasiou (2019)]

$$\cdots \otimes ~~s_{345} \otimes s_{234}~~ \otimes \cdots$$

Adjacency Constraints

adjacency constraints	which singularities are kinematically compatible with each other?	$\cdots \otimes \bullet \otimes \bullet \otimes \bullet \otimes \cdots$

- Recent progress has also been made understanding the (non-extended) Steinmann relations

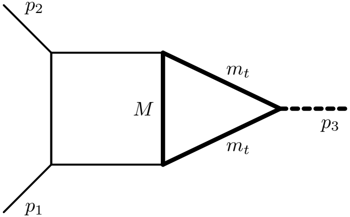


- Namely, the Steinmann relations are (conjecturally) expected to be violated in two-loop integrals only when one of the above five cuts is present [Hannesdottir, Lippstreu, **AJM**, Polackova (2025)]
- This means that we can assume the Steinmann relations hold whenever these cuts are absent

Adjacency Constraints

adjacency constraints	which singularities are kinematically compatible with each other?	$\cdots \otimes \bullet \otimes \bullet \otimes \bullet \otimes \cdots$

- Additional adjacency relations have been observed in all but the simplest Feynman integrals



$$\not\supset \quad \cdots \otimes \big(s - 4m_t^2\big) \otimes \big(s - (m_t + M)^2\big) \otimes \cdots$$

$$s = (p_1 + p_2)^2$$

- However, no practical method is yet known for predicting these diagram-specific relations, meaning we **can currently only leverage universal constraints in bootstrap calculations**

The Hierarchical Principle

hierarchical principle	which singularities arise from the same on-shell physics?	$\cdots \otimes \bullet \otimes \cdots \otimes \bullet \otimes \cdots$

What happens after we start computing discontinuities?

$$\mathcal{I}_G \qquad \longrightarrow \qquad \text{Disc}_\lambda \mathcal{I}_G$$

- **Can all of the singularities of the original integral still arise** in this discontinuity?

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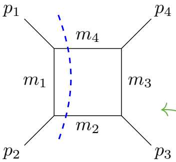
What happens after we start computing discontinuities?

$$\mathcal{I}_G \qquad \longrightarrow \qquad \text{Disc}_\lambda \mathcal{I}_G$$

- **Can all of the singularities of the original integral still arise** in this discontinuity?
- **No!** This can already be seen in simple examples via Cutkosky's formula [Cutkosky (1960)]



$$\longrightarrow \qquad \text{Disc}_{s-(m_2+m_4)^2} \mathcal{I}_{\square} \propto \int d^D\ell \, \frac{\delta^+(q_2^2-m_2^2)\delta^+(q_4^2-m_4^2)}{(q_1^2-m_1^2)(q_3^2-m_3^2)}$$

$$s = (p_1+p_2)^2$$

a singularity in which the propagators with mass m_2 and m_4 pinch the contour

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More generally, the **hierarchical principle** states that propagators that are put on shell when we compute discontinuities must stay on shell [Landshoff, Olive, Polkinghorne (1965)] [Pham (1967)] [Boyling (1968)]

- This suggests that a **modified set of Landau equations** should apply to $\text{Disc}_\lambda \mathcal{I}_G$:

$q_e^2 - m_e^2 = 0$	for every edge e involved in pinching the contour when $\lambda = 0$
$\alpha_e(q_e^2 - m_e^2) = 0$	for every edge e that did not participate in the pinch
$\sum_{e \in \text{loop}} \alpha_e q_e^\mu = 0$	for each independent loop in G

- If $\lambda' = 0$ no longer appears as a solution to these modified equations, we have

$$\text{Disc}_{\lambda'} \text{Disc}_\lambda \mathcal{I}_G = 0$$

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- Although this principle goes back over fifty years, a practical algorithm for deriving hierarchical constraints has only recently been devised [Hannesdottir, Lippstreu, **AJM**, Polackova (2024)]

- Essential, this involves tracking how discontinuities modify the $\alpha_e = 0$ integration endpoints, and checking which singularities no longer arise once these endpoints are absent [Britto (2023)] [Fevola, Mizera, Telen (2023)]

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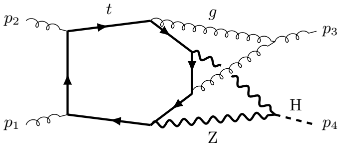
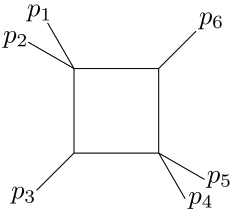
- This can be done efficiently, even for complicated examples



$$\not\supset \quad \cdots \otimes \left(m_H^2 - 4m_Z^2\right) \otimes \cdots \otimes \left(s_{12} + s_{23} - m_H^2 + m_Z^2\right) \otimes \cdots$$

Example: the Two-Mass Box Integral



s_{12} $\quad$ s_{123} $\quad$ s_{345} $\quad$ s_{45}

$s_{12} - s_{123}$ $\quad$ $s_{12} - s_{345}$ $\quad$ $s_{45} - s_{123}$ $\quad$ $s_{45} - s_{345}$

$s_{12}s_{45} - s_{123}s_{345}$ $\quad$ $s_{12} + s_{45} - s_{123} - s_{345}$

- This integral depends on four Mandelstam variables, and involves ten symbol letters

$$s_{ij} = (p_i + p_j)^2 \qquad s_{ijk} = (p_i + p_j + p_k)^2$$

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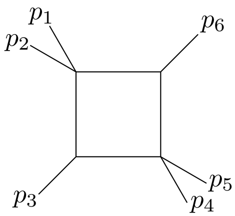

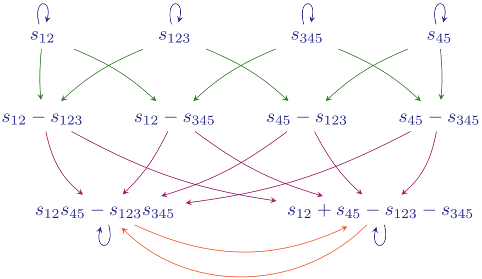

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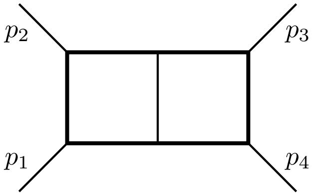
- Applying these new methods, we are able to derive 68 hierarchical constraints of the form
  [Hannesdottir, Lippstreu, **AJM**, Polackova (2024)] [Crisanti, Lippstreu, **AJM**, Polackova (to appear)]

$$\mathsf{Disc}_{\lambda'} \cdots \mathsf{Disc}_{\lambda} \cdots \mathcal{I}_{\boxtimes} = 0$$

- These constraints are indeed obeyed (to all orders in dimensional regularization)

Example: the Double Box with Internal Mass

Let's now see how this all comes together in a two-loop example, involving four particles and an internal mass m [Hannesdottir, **AJM**, Schwartz, Vergu (2024)]



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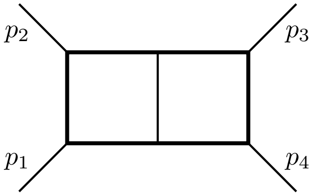

$$s = (p_1 + p_2)^2$$

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- We first solve the **Landau equations**, finding five α-positive solutions:

$$s = 4m^2, \qquad s \to \infty, \qquad t = 4m^2, \qquad t \to \infty, \qquad m^2 = 0$$

and five more non-α-positive solutions:

$$s = 0, \qquad t = 0, \qquad m^2 \to \infty, \qquad s + t = 0, \qquad st - 4m^2 s - 4m^2 t = 0$$

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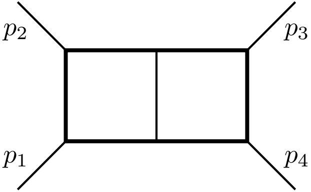

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- One can construct **12 algebraic symbol letters** that are consistent with these singularities [Heller, von Manteuffel, Schabinger (2019)]

Example: the Double Box with Internal Mass

Constraint	Undetermined Coefficients
Sequences of four letters	20736
Partial derivatives commute	6993

possible symbol terms

$$\mathcal{I}_{\text{ansatz}} = \sum c_i S_i$$

undetermined coefficients

- To construct our initial ansatz, we build the **complete space of weight-four polylogarithmic symbols** that can be constructed using these letters
- To make sure all of the symbols we consider correspond to well-defined functions, we require **all sequences of partial derivatives acting on $\mathcal{I}_{\text{ansatz}}$ commute**

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- Knowing which singularities arise are **α-positive**, we can restrict which discontinuities appear on the principle Riemann sheet

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- Knowing which singularities arise are **α-positive**, we can restrict which discontinuities appear on the principle Riemann sheet
- The **hierarchical principle** implies the further constraints:

$$\text{Disc}_{t-m^2} \cdots \text{Disc}_X \cdots \mathcal{I}_{\text{ansatz}} = 0$$

$$X \in \left\{ s\,,\; s - m^2\,,\; s + t\,,\; \frac{1}{s}\,,\; st - 4m^2 s - 4m^2 t \right\}$$

Example: the Double Box with Internal Mass

Constraint	Undetermined Coefficients
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Only square root branch cuts at $s = 4m^2$ in physical region	6
Only square root branch cuts at $t = 4m^2$	1

- Finally, by studying the nature of the s and t threshold singularities, we can place constraints on **what types of branch cuts appear** on these singular surfaces

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- Finally, by studying the nature of the s and t threshold singularities, we can place constraints on **what types of branch cuts appear** on these singular surfaces
- We can determine the value of the final coefficient by comparing to the **leading singularity**

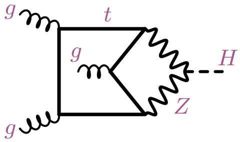
We manage to reproduce the correct result for this Feynman integral
just from knowledge of its singularity structure

[Caron-Huot, Henn (2014)]

Conclusions and Outlook

Building on the success of **perturbative bootstrap methods** in planar $\mathcal{N} = 4$, we have shown that Feynman integrals can also be computed just from **knowledge of their singular behavior**

- This approach builds on many recent advances in our understanding of the analytic structure of Feynman integrals and amplitude, for instance with respect to the **hierarchical principle**
- Similar breakthroughs for **α-positivity** and **adjacency constraints** look within reach
- These methods look especially promising when many masses are involved — for instance, in **mixed QCD-electroweak corrections in the Standard Model**



Can these ideas also be applied to full amplitudes—especially **Standard Model amplitudes**?

Thanks!